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ABSTRACT

In this paper, mechanisms that differentiate microrotation (i.e., particles' internal rotation) and vorticity in magnetohydrodynamic micropolar flows with magnetic particles are examined. Micropolar fluids are characterized by the asynchronization of the microrotation and the vorticity of the liquid carrier. When the microrotation is equal to the fluid's vorticity, the flow is purely Newtonian. In this context, two classical examples of micropolar magnetohydrodynamic fluids with magnetic particles are used, namely, blood and ferrofluid. The effect of all dimensionless parameters associated with the mathematical model is examined in terms of differentiating microrotation and vorticity. These dimensionless parameters are specifically calculated using the physical properties of the ferrofluid and blood. It was shown that higher values of the rotational viscosity (which are associated with smaller values of the spin relaxation time), higher values of channel's height, and higher values of the microrotation wall parameter tend to equalize microrotation and vorticity. On the other hand, the spin viscosity and the micromagnetorotation (magnetic torque) are mechanisms that differentiate microrotation and vorticity. Lorentz force does not seem to have any noticeable effect on the microrotation–vorticity difference. It is anticipated that this study will reveal the cases where a magnetohydrodynamic micropolar fluid with magnetic particles, such as blood and ferrofluid, can be simplified to a Newtonian one, which brings many benefits associated with a simplified mathematical flow model (such as smaller computational cost and less time).

NOMENCLATURE

$\boldsymbol{A}$	Antisymmetric part of the stress tensor $\boldsymbol{T}$ $(\mathrm{kg/ms^2})$
a	Particles' radius (m)
$\boldsymbol{B}$	Magnetic flux density (T)
B_0	Flux density of the applied magnetic field (T)
$\boldsymbol{C}$	Couple stress tensor $(\mathrm{kg/s^2})$
$c_0,\ c_\alpha,\ c_d$	Angular viscosity coefficients (kgm/s)
$\boldsymbol{f}$	External (body) force per unit mass $(\mathrm{m/s^2})$
$\boldsymbol{G}$	Body torque per unit mass $(\mathrm{m/s^2})$
G	Pressure gradient constant $(\mathrm{kg/m^2s^2})$
$\boldsymbol{H}$	Vector of the applied magnetic field's intensity (A/m)
H	Magnitude of the applied magnetic field's intensity (A/m)
H_0	Applied magnetic field's intensity (A/m)
h	Ratio of magnetization to applied magnetic field $(-)$
I	Sum of the particles' moment of inertia per unit volume (kg/m)
$\boldsymbol{j}$	Current density vector $(\mathrm{A/m^2})$
j	Microinertia coefficient $(\mathrm{m^2})$
k_β	Boltzmann constant $(\mathrm{kg\,m^2/s^2K})$
$\boldsymbol{L}$	Total angular momentum per unit mass $(\mathrm{m^2})$
L	Channel's half-width (m)
$L(\zeta)$	Langevin function $(-)$
$\boldsymbol{l}$	Internal angular momentum per unit mass $(\mathrm{m/s^2})$
$\boldsymbol{M}$	Magnetization vector (T)
M_0	Equilibrium magnetization (T)
M_s	Saturation magnetization (T)
m	Magnetic dipole moment $(\mathrm{Am^2})$
N	Particles' number per unit volume $(\mathrm{m^{-3}})$

p	Pressure $\left(\text{kg}/\text{ms}^2\right)$
Re_m	Magnetic Reynolds number $(-)$
$\boldsymbol{r}$	Position vector (m)
$\boldsymbol{T}$(or T_{ij})	Stress tensor $\left(\text{kg}/\text{ms}^2\right)$
T	Absolute temperature (K)
t	Time (s)
V_0	Velocity of the upper plane (m/s)
x, y, z	Coordinate variables (m)

Greek symbols

α, β	Angular viscosity coefficients (kg m/s)
γ	Spin viscosity (kg m/s)
δ	Microrotation boundary condition parameter $(-)$
δ_{ij}	Kronecker's delta $(-)$
ϵ_{mij}	Levi-Civita symbol $(-)$
μ	Dynamic viscosity (kg /ms)
μ_B	Bulk viscosity (kg /ms)
μ_r	Rotational viscosity (kg /ms)
μ_0	Magnetic permeability of free space ($4\pi \times 10^{-7}$ H/m)
υ	Velocity (m/s)
ρ	Fluid's density $\left(\text{kg}/\text{m}^3\right)$
ρ_a	Particles' density $\left(\text{kg}/\text{m}^3\right)$
σ	Electrical conductivity $\left(\text{kgm}^3/\text{A}^2\,\text{s}^3\right)$
τ	Magnetization relaxation time (s)
τ_B	Brownian time of rotational diffusion (s)
τ_s	Spin relaxation time (s)
φ	Volume fraction of the dispersed phase $(-)$
χ_m	Magnetic susceptibility (H/m)
$\boldsymbol{\Omega}$	Vorticity (rad/s)
$\boldsymbol{\omega}$	Microrotation (rad/s)

Subscripts and superscripts

x, y, z	Component in the x, y, z direction

I. INTRODUCTION

Micropolar fluids are fluids with internal microstructure. In general, they are considered as stable suspensions consisting of rigid, randomly oriented, spherical particles in a mother-liquid carrier.[1] These particles exhibit internal degrees of freedom that seem to play an important role in establishing an equilibrium macroscopic thermodynamic theory for micropolar fluids.[2] The micropolar fluid theory, as established by Eringen,[3,4] is an extension of the classical model for Newtonian fluids i.e., the Navier–Stokes equations.[5] Some characteristic examples of micropolar fluids are lubricants,[6] colloidal suspensions (such as ferrofluids),[7] liquid crystals,[8] and blood.[9] The main feature of micropolar fluids is microrotation, a pseudovector that represents the particles' average rotational velocity.[10]

The history of micropolar fluid theory goes back to the establishment of the antisymmetric stress tensor using classical irreversible thermodynamics. In their well-known book, DeGroot and Mazur[11] derived a constitutive equation for the antisymmetric stress tensor, which contributes to the change of the internal angular momentum. Imagine a line of flowing fluid with suspended particles that rotates with vorticity $\boldsymbol{\Omega} = \frac{1}{2}\nabla \times \upsilon$. In equilibrium steady states, the suspended particles will tend to rotate with the same angular velocity as the fluid's vorticity (i.e., microrotation $\boldsymbol{\omega}$ is equal to vorticity, $\boldsymbol{\Omega} = \boldsymbol{\omega}$) in spin relaxation time τ_s. In other circumstances, friction forces arise between microrotation and vorticity, which result in asynchronization between the latter. Such friction forces are a function of the microrotation–vorticity difference $\nabla \times \upsilon - 2\boldsymbol{\omega}$. In this manner, the antisymmetric part of the stress tensor is exclusively depended on this difference. Here, the well-known rotational viscosity μ_r is introduced as an integral part of the antisymmetric stress tensor related to microrotation–vorticity difference and the spin relaxation time.[12]

In the book of DeGroot and Mazur,[11] the flux of the internal angular momentum related to the couple stress tensor is ignored. Eringen used irreversible thermodynamics to derive a constitutive equation for the couple stress tensor including the work of DeGroot and Mazur for the antisymmetric part of the stress tensor. His work is considered as the final version of the micropolar fluid theory. There are also other studies considering fluids with internal microstructure introducing different versions of the couple stress tensor such as Condiff and Dahler,[2] Aero *et al.*,[13] and Shliomis.[14] As a result of the addition of the couple stress tensor, a new type of viscosity is introduced, namely, the spin viscosity.

Ferrofluids are an important class of micropolar fluids, since they are made of ferromagnetic nanoparticles (such as magnetite) into a mother-liquid carrier (such as water or kerosene). Ferrofluids can have applications in both engineering and biomedicine, mainly because of their rheological properties that can be easily controlled by an externally applied magnetic field.[7,15] Ferrohydrodynamics deals with the mechanics of the ferrofluids motion. The first systematic study of ferrohydrodynamics was made by Rosensweig[16] in his famous book. As expected, ferrofluids exhibit rotational degrees of freedom associated with the rotational motion of the suspended ferromagnetic particles. The rotational motion of the ferromagnetic particles is treated exactly as the microrotation in other micropolar fluids.

Another important issue in ferrofluids is the magnetic relaxation equation. The ferromagnetic particles exhibit strong forces because of the magnetic polarization (magnetization) when an external magnetic field is applied. In general, the magnetization field $\boldsymbol{M}$ can relax with the magnetic field $\boldsymbol{H}$ in relaxation time τ. Due to this misalignment, a magnetic torque $\boldsymbol{M} \times \boldsymbol{H}$ arises, which also affects the microrotation of the ferromagnetic particles. Shliomis was the first to notice that the magnetic torque acts as an external force, which differentiates the microrotation of the ferromagnetic particles from the vorticity of the mother-liquid carrier. He also found that any difference between the microrotation and the vorticity gives rise to dissipation processes that lead to an increase in the effective viscosity.[17–19] The work of Shliomis has been utilized by many researchers for studying ferromagnetic flows, such as the classical ferrofluid Couette–Poiseuille flow by Yang and Liu,[20] the ferrofluid cavity flow by Singh *et al.*,[21] the ferromagnetic blood flow by Pai *et al.*,[22] and the homogenous ferrofluid turbulence under the influence of a steady magnetic field by Schumacher *et al.*[23]

The main difference of the ferrohydrodynamic model derived by Rosensweig[16] and Shliomis[17–19] and the micropolar fluid theory is the absence of the couple stress tensor. The stress tensor used in this ferrohydrodynamic model is antisymmetric and identical to the one of DeGroot and Mazur,[11] based only on the rotational viscosity and the vorticity–microrotation difference. These researchers also hypothesized that the mother-liquid carrier of the ferrofluid in non-conducting and the Lorentz force has no effect on the flow. Moreover, Shliomis[24] hypothesized that in the absence of an external magnetic field, the ferrofluid's suspended particles are only influenced by the friction forces exerted by the mother liquid, which are not enough to maintain a difference between microrotation and vorticity. In this case, the spin relaxation time is very small ($\tau_s \lesssim 10^{-8}$), which means that the equalization of microrotation and vorticity happens very quickly and any differences between the latter are not noticeable. Then, the antisymmetric part of the stress tensor can be ignored and the ferrofluid flow is described by the classical model of the Navier–Stokes equations. He also showed that the spin viscosity is extremely small, which means that the effect of the couple stress tensor on a ferrofluid can be ignored.

Lately, Shizawa and Tanahashi[25] and Rosensweig[26] developed a model for conducting micropolar ferrofluids using irreversible thermodynamics. The couple stress tensor that they derived is the same with the one of the micropolar fluid theory. They also included the effect of the Lorentz force in the law of the conservation of linear momentum. The constitutive magnetization equation derived by Shizawa and Tanahashi[25] coincide with the relaxation magnetization equation derived by Rosensweig[26] and Shliomis[17] in steady-state conditions. In this manner, the mathematical model of Shizawa and Tanahashi is suitable for exploring the mechanisms that affect the microrotation and vorticity in order to fully identify the cases that a ferrofluid can be simplified to a Newtonian fluid. To the authors' knowledge, there are only a few studies that use the of theory of Shizawa and Tanahashi.[25] Recently, Aslani *et al.*[10,27,28] made a number of analytical studies on the impact of magnetic torque (micromagnetorotation) on micropolar magnetohydrodynamic (MHD) flows with the use of Shizawa and Tanahashi model. None of these studies dealt with the mechanisms that affect the microrotation–vorticity difference.

Blood is also considered a micropolar fluid by many researchers, due to the existence of the blood cells in the plasma.[29–33] The experimental study of Ariman *et al.*[34] on steady and pulsatile blood flows showed a good agreement with the results that micropolar fluid theory predicts for these flows. There are also studies which suggest that blood should be treated as a ferrofluid when it is under the influence of an external applied magnetic field (for example, when a magnetic resonance image (MRI) scanner is used). Blood contains the hemoglobin molecule in the erythrocytes (red blood cells), which is an iron oxide and behaves like a magnetic particle, while blood plasma is the mother-liquid carrier.[35,36] In this manner, it is possible that the applied magnetic field affects the erythrocytes microrotation due to hemoglobin's magnetization, which in turn affects blood's viscosity and velocity.[37] There are experimental studies that show statistically significant evidence of symptoms, such as vertigo, nausea, and metallic taste for magnetic field intensities of 1.5 and 4 T associated with blood's velocity reduction.[38] Moreover, these findings cannot be explained by the effect of the Lorentz force on the blood flow, as the Lorentz force cannot affect blood due to the latter's small electrical conductivity.[39] To the authors' knowledge, despite the evidence, the consideration of blood as a ferrofluid under the influence of an applied magnetic field is not widely accepted and blood is examined as a classical MHD Newtonian fluid.[40,41]

It is obvious that there is a gray area regarding the cases that a micropolar MHD fluid with magnetic particles can be simplified to the corresponding Newtonian. For this reason, mechanisms that differentiate microrotation and vorticity (this difference is very important for the consideration of a fluid as micropolar) in MHD micropolar flows with magnetic particles are examined. Two classical examples of micropolar MHD fluids with magnetic particles are used, namely, blood and ferrofluid. These fluids are modeled as 2D Couette and Poiseuille flows. The effect of all dimensionless parameters, which are relevant with the mathematical model of Shizawa and Tanahashi,[25] was examined in terms of differentiating $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$. These dimensionless parameters were specifically calculated using the physical properties of the ferrofluid and blood. In this paper, it anticipated to identify when an MHD micropolar fluid with magnetic particles, such as blood and ferrofluid, can be simplified to a Newtonian one, a situation which brings many benefits associated with a simplified mathematical flow model (such smaller computational cost and less time). Moreover, the full effect of the rotational and spin viscosities along with the micromagnetorotation term (which, as discussed above, has been studied only by a very small number of papers) on ferrofluid and blood is examined and discussed in terms of differentiating microrotation and vorticity.

II. MATHEMATICAL PRELIMINARIES

A. Micropolar fluid theory

As it is mentioned above, the micropolar fluid theory is based on the fact that the solid, randomly oriented particles suspended in a viscous medium can, in principle, rotate relative to the mother-liquid. When the particles' rotational velocity, i.e., the microrotation $\boldsymbol{\omega}$ does not coincide with the local angular velocity of the mother-liquid, i.e., $\boldsymbol{\Omega} = \frac{1}{2}\nabla \times \boldsymbol{v}$, the stress tensor becomes antisymmetric. For a Newtonian fluid, the well-known Cauchy's equation of motion (conservation of linear momentum) is written in differential form as

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \boldsymbol{f} + \nabla \cdot \boldsymbol{T}, \tag{1}$$

where $\boldsymbol{f}$ represents external or body forces (per unit mass) such as gravity or electromagnetic forces applied on the fluid and $\boldsymbol{T} = T_{ij}$ is the stress s tensor. In classical Newtonian hydrodynamics, the stress tensor is assumed to be symmetric, i.e., $T_{ij} = T_{ji}$, where $i, j = 1, 2, 3$. This argument is based on the law of conservation of angular momentum, which is directly derived from the Cauchy's equation of motion and does not allow any internal rotations. The proof of this assumption can be found in many textbooks, such as the ones of Lukaszewicz[1] and DeGroot and Mazur.[11] When internal rotations exist, the conservation of the total angular momentum $\boldsymbol{L}$ (per unit mass) in a closed volume of fluid is written as follows:

$$\frac{D}{Dt}\iiint \rho \boldsymbol{L} dV = \iiint \rho(\boldsymbol{r} \times \boldsymbol{f} + \boldsymbol{G}) dV + \oiint \hat{\boldsymbol{n}} \cdot (\boldsymbol{C} + \boldsymbol{r} \times \boldsymbol{T})\, dS, \tag{2}$$

where $\boldsymbol{r} = (x, y, z)$ is the position vector. Moreover, $\boldsymbol{G}$ is the body torque per unit mass in addition to the external body force per unit mass $\boldsymbol{f}$ and $\boldsymbol{C}$ is the couple stress tensor in addition to the stress tensor $\boldsymbol{T}$. Using Gauss' divergence theorem, Eq. (2) can be re-written as follows:

$$\frac{D}{Dt}\iiint \rho \boldsymbol{L} dV = \iiint (\rho \boldsymbol{r} \times \boldsymbol{f} + \rho \boldsymbol{G} + \nabla \cdot \boldsymbol{C} + \boldsymbol{r} \times \nabla \cdot \boldsymbol{T} + \boldsymbol{A}) dV. \tag{3}$$

Here, $\boldsymbol{A}$ is the antisymmetric part of the stress tensor $\boldsymbol{T}$ with components $T_{23} - T_{32}$, $T_{31} - T_{13}$ and $T_{12} - T_{21}$. It should be noted that when the stress tensor $\boldsymbol{T}$ is symmetric (as it is in the classical Newtonian hydrodynamics), then $T_{ij} = T_{ji}$ and $\boldsymbol{A} = 0$. By applying Reynolds' transport theorem on Eq. (3), the conservation of the total angular momentum is written in differential form as

$$\rho \frac{D\boldsymbol{L}}{Dt} = \rho(\boldsymbol{r} \times \boldsymbol{f} + \boldsymbol{G}) + \nabla \cdot \boldsymbol{C} + \boldsymbol{r} \times (\nabla \cdot \boldsymbol{T}) + \boldsymbol{A}. \tag{4}$$

The total angular momentum $\boldsymbol{L}$ (per unit mass) is the sum of the moment of linear momentum $\boldsymbol{r} \times \boldsymbol{u}$ (or the external angular momentum per unit mass) and the internal angular momentum (per unit mass) $\boldsymbol{l}$ as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{u} + \boldsymbol{l}. \tag{5}$$

The internal angular momentum (per unit mass) $\boldsymbol{l}$ is interpreted by the rotation of the particles in the viscous medium. When the vector product of $\boldsymbol{r}$ and Cauchy's equation of motion [Eq. (1)] is subtracted from Eq. (4) while using Eq. (5), it is derived that

$$\rho \frac{D\boldsymbol{l}}{Dt} = \rho \boldsymbol{G} + \nabla \cdot \boldsymbol{C} + \boldsymbol{A}. \tag{6}$$

Equation (6) is the equation of change of the internal angular momentum; it is an important part of the micropolar fluid theory. From this equation and Eq. (4), it should be noted that neither the external angular momentum nor the internal angular momentum are conserved. There is a loss of external angular momentum represented by the antisymmetric part of the stress tensor $\boldsymbol{A}$ that is found as a gain in the internal angular momentum. This mechanism is directly related to the difference between the microrotation of the particles and the local angular velocity of the mother-liquid $\boldsymbol{\Omega} - \boldsymbol{\omega}$. It is assumed that the internal angular momentum (per unit mass) $\boldsymbol{l}$ can be written as a vector with three components as

$$\boldsymbol{l} = j\boldsymbol{\omega}, \tag{7}$$

where j is a scalar quantity, namely, the microinertia coefficient. A new scalar quantity can be also introduced, namely, the sum of the particles' moment of inertia per unit volume $I = \rho j$. The microinertia coefficient can be calculated as $j = \frac{8}{15}\pi a^5 N \frac{\rho_a}{\rho}$, where N, a, and ρ_a are the number (per unit volume), radius, and density of the suspended particles, respectively.[18,25,42] Using the definition of the volume fraction $\varphi = \frac{4}{3}\pi \alpha^3 N$,[18,25,42] it is derived that $j = \frac{2}{5}\alpha^2 \frac{\rho_a}{\rho}\varphi$. Using Eq. (7), Eq. (6) can be written as follows:

$$\rho j \frac{d\boldsymbol{\omega}}{dt} = \rho \boldsymbol{G} + \nabla \cdot \boldsymbol{C} + \boldsymbol{A}. \tag{8}$$

In order to fully derive the governing equations of the micropolar fluid theory, the stress tensor $\boldsymbol{T}$ and the couple stress tensor $\boldsymbol{C}$ should be introduced. The term $\nabla \cdot \boldsymbol{T}$ corresponds to the flux of the linear momentum, while $\nabla \cdot \boldsymbol{C}$ corresponds to the flux of the intrinsic angular momentum. As mentioned in the Introduction, many researchers have calculated constitutive relations for $\boldsymbol{C}$. According to Lukaszewicz,[1] the stress tensor $\boldsymbol{T}$ for fluids with internal rotations is written as follows:

$$T_{ij} = (-p + \mu_B \upsilon_{k,k})\delta_{ij} + \mu(\upsilon_{i,j} + \upsilon_{j,i}) + \mu_r(\upsilon_{j,i} - \upsilon_{i,j}) - 2\mu_r \epsilon_{mij}\omega_m, \tag{9}$$

where μ and μ_B are the dynamic and bulk viscosities, respectively, as they are introduced in the classical Newtonian hydrodynamics, μ_r is the rotational or dynamic microrotational viscosity, and δ_{ij} is Kronecker's delta. The stress tensor derived by Lukaszewicz[1] is identical to the one derived by Eringen[4] for $\mu = \mu_\nu + \frac{k_\nu}{2}$ and $\mu_r = \frac{k_\nu}{2}$. It should be also noted that the symmetric part of this stress tensor is

$$T_{ij}^{(s)} = (-p + \mu_B \upsilon_{k,k})\delta_{ij} + \mu(\upsilon_{i,j} + \upsilon_{j,i}), \tag{10}$$

which is identical with the stress tensor of a Newtonian fluid. According to Shliomis,[14,17,24] T_{ij} can be expressed as

$$T_{ij} = -p\delta_{ij} + \mu(\upsilon_{i,j} + \upsilon_{j,i}) - \frac{1}{2\tau_s}\left(l_{ij} - I\Omega_{ij}\right), \tag{11}$$

where τ_s is the spin relaxation time. The internal angular momentum l_{ij} and the local angular momentum Ω_{ij} are defined as

$$\Omega_{ij} = \frac{1}{2}(\upsilon_{j,i} - \upsilon_{i,j}) = \epsilon_{mij}\Omega_m, \tag{12}$$

$$l_{ij} = \epsilon_{mij} l_m = I\epsilon_{mij}\omega_m, \tag{13}$$

where ϵ_{mij} is the Levi-Civita symbol $(m, i, j = 1, 2, 3)$. It should be noted that Shliomis[17,18] uses angular momentum per unit volume for his governing equations unlike Lukaszewicz,[1] DeGroot and Mazur,[11] and Shizawa and Tanahashi[25,42] who use angular momentum per unit mass. Inserting Eqs. (12) and (13) into Eq. (11), it is evident that the two constitutive equations for the stress tensor [Eqs. (9) and (11)] are identical (excluding the bulk viscosity term $\mu_B \upsilon_{k,k}$). This implies that the rotational viscosity μ_r is defined as

$$\mu_r = \frac{I}{4\tau_s}, \tag{14}$$

where $\tau_s = \frac{\alpha^2 \rho_\alpha}{15\mu}$. This relation for the rotational viscosity μ_r [Eq. (14)] is also reported by DeGroot and Mazur[11] and Shizawa and Tanahashi.[25,42,43] Interestingly, many famous textbooks for micropolar fluids (see Eringen [3,4] and Lukaszewicz[1]) do not mention that Eq. (14) is valid for the rotational viscosity μ_r. Lukaszewicz[1] also introduced a constitutive equation for $\boldsymbol{C}$ as

$$C_{ij} = c_0 \omega_{k,k}\delta_{ij} + c_d(\omega_{i,j} + \omega_{j,i}) + c_\alpha(\omega_{j,i} - \omega_{i,j}), \tag{15}$$

where c_0, c_d, and c_α are angular viscosities. This couple stress tensor is also identical to the one derived by Eringen[4] $C_{ij} = \alpha\omega_{k,k}\delta_{ij} + \beta\omega_{i,j} + \gamma\omega_{j,i}$ for $c_0 = \alpha$, $c_d - c_\alpha = \beta$, and $c_d + c_\alpha = \gamma$. As mentioned in Introduction, Shliomis[14] has also derived a constitutive equation for $\boldsymbol{C}$, which is equal to the third term of Eringen's couple stress tensor, i.e., $C_{ij} = \gamma\omega_{j,i}$.

Inserting Eqs. (9) and (15) into Eqs. (1) and (8), the governing equations for micropolar fluids are written as[1]

$$\rho \frac{D\boldsymbol{\upsilon}}{dt} = -\nabla p + (\mu_B + \mu - \mu_r)\nabla(\nabla \cdot \boldsymbol{\upsilon}) + (\mu + \mu_r)\nabla^2 \boldsymbol{\upsilon} + 2\mu_r \nabla \times \boldsymbol{\omega} + \rho \boldsymbol{f}, \tag{16}$$

$$\rho j \frac{D\boldsymbol{\omega}}{Dt} = 2\mu_r(\nabla \times \boldsymbol{\upsilon} - 2\boldsymbol{\omega}) + (c_0 + c_d - c_\alpha)\nabla(\nabla \cdot \boldsymbol{\omega}) + \gamma \nabla^2 \boldsymbol{\omega} + \rho \boldsymbol{G}. \tag{17}$$

One equation should be also added in these governing equations system, namely, the law of conservation of mass

$$\frac{D\rho}{Dt} = -\rho(\nabla \cdot \boldsymbol{v}). \tag{18}$$

Provided that the micropolar fluid is viscous and incompressible, Eq. (18) becomes

$$\nabla \cdot \boldsymbol{v} = 0. \tag{19}$$

Equation (19) also implies that

$$\nabla \cdot \boldsymbol{\Omega} = 0. \tag{20}$$

Using Eqs. (19) and (20), Eqs. (16) and (17) become

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + (\mu + \mu_r)\nabla^2\boldsymbol{v} + 2\mu_r\nabla \times \boldsymbol{\omega} + \rho\boldsymbol{f}, \tag{21}$$

$$\rho j\frac{D\boldsymbol{\omega}}{Dt} = 2\mu_r(\nabla \times \boldsymbol{v} - 2\boldsymbol{\omega}) + \gamma\nabla^2\boldsymbol{\omega} + \rho\boldsymbol{G}. \tag{22}$$

As mentioned above, $\mu_r = \frac{I}{4\tau_s}$ and $\gamma = c_d + c_\alpha$. As reported by Shizawa and Tanahashi,[25,42,43] Shliomis,[24] and Allen and Kline,[44] the spin viscosity γ equals to

$$\gamma = \mu j \quad \text{or} \quad \gamma = \mu\frac{I}{\rho}. \tag{23}$$

Moreover, the second law of thermodynamics requires that

$$\mu \geq 0, \quad \mu_r \geq 0, \quad \gamma \geq 0. \tag{24}$$

Using the definition of vorticity $\boldsymbol{\Omega} = \frac{1}{2}\nabla \times \boldsymbol{v}$, Eqs. (21) and (22) can be rewritten as

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + \mu\nabla^2\boldsymbol{v} + 2\mu_r\nabla \times (\boldsymbol{\omega} - \boldsymbol{\Omega}) + \rho\boldsymbol{f}, \tag{25}$$

$$\rho j\frac{D\boldsymbol{\omega}}{Dt} = 4\mu_r(\boldsymbol{\Omega} - \boldsymbol{\omega}) + \gamma\nabla^2\boldsymbol{\omega} + \rho\boldsymbol{G}. \tag{26}$$

Both forms used for the micropolar governing equations [Eqs. (21) and (22) and (25) and (26)] are equivalent. It is evident that when $\mu_r = 0$ or when $\boldsymbol{\omega} = \boldsymbol{\Omega}$, the classical Newtonian hydrodynamic equations are retrieved.

B. Ferrohydrodynamics

As it is mentioned in Introduction, the motion of ferrofluids can be examined with the use of the micropolar fluid theory. Many researchers have established mathematical models for ferrohydrodynamics based on the rotational motion of the suspended particles relative to the mother-liquid, such as Shliomis,[17,18] Shizawa and Tanahashi,[25,42] and Rosensweig.[16,26] Essentially, two terms should be added in the micropolar governing equations [Eqs. (25) and (26)], i.e., a magnetic body force in the equation of conservation of the linear momentum and a magnetic torque in the equation of change of the internal angular momentum that acts directly upon the magnetic particles' rotation. In this manner, Eqs. (25) and (26) are written as follows:[16,17,24,42]

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + \mu\nabla^2\boldsymbol{v} + 2\mu_r\nabla \times (\boldsymbol{\omega} - \boldsymbol{\Omega}) + (\boldsymbol{M} \cdot \nabla)\boldsymbol{H}, \tag{27}$$

$$\rho j\frac{D\boldsymbol{\omega}}{Dt} = 4\mu_r(\boldsymbol{\Omega} - \boldsymbol{\omega}) + \gamma\nabla^2\boldsymbol{\omega} + \boldsymbol{M} \times \boldsymbol{H}. \tag{28}$$

The magnetization vector $\boldsymbol{M}$ and the applied magnetic field $\boldsymbol{H}$ form the magnetic flux density $\boldsymbol{B} = \mu_0\boldsymbol{H} + \boldsymbol{M}$, where μ_0 is the magnetic permeability of free space ($4\pi \times 10^{-7}\ \frac{\mathrm{H}}{\mathrm{m}}$). The stress tensor for ferrofluids consists of two parts, the viscous part as it is written in Eq. (9) and the magnetic part (Maxwell tensor):[17,24,25,42]

$$T_{ij} = (-p + \mu_B v_{k,k})\delta_{ij} + \mu(v_{i,j} + v_{j,i}) + 2\mu_r\epsilon_{mij}(\Omega_m - \omega_m) + B_iH_j - (\mu_0H^2/2)\delta_{ij}. \tag{29}$$

The couple stress tensor [Eq. (15)] stays the same. The Maxwell's equations are also needed

$$\nabla \cdot \boldsymbol{B} = 0, \tag{30}$$

$$\nabla \times \boldsymbol{H} = 0. \tag{31}$$

Equations (27) and (28) contain the magnetization vector $\boldsymbol{M}$ as unknown, which implies that an additional equation is required to make the system determinate. The magnetization influences the fluid motion, especially the rotation of the particles and depends itself on this rotation as well. The relaxation equation of magnetization as it is introduced by Shliomis[17,18] and Rosensweig[16,26] is used

$$\frac{D\boldsymbol{M}}{Dt} = \boldsymbol{\omega} \times \boldsymbol{M} - \frac{1}{\tau_B}\left(\boldsymbol{M} - M_0\frac{\boldsymbol{H}}{H}\right), \tag{32}$$

where H the magnitude of the applied magnetic field's intensity, M_0 is the equilibrium magnetization, and τ_B is the Brownian time of rotational diffusion. The equilibrium magnetization M_0 is defined as the magnetization attained by the magnetic particles in stationary state, when a steady magnetic field $\boldsymbol{H}$ is applied. In general, the equilibrium magnetization tends to orient in the direction of the applied magnetic field. It can be calculated as follows:

$$M_0 = \varphi M_s L(\zeta) = \varphi M_s\left(\coth\zeta - \frac{1}{\zeta}\right), \quad \zeta = \frac{\mu_0 mH}{k_\beta T}, \tag{33}$$

where M_s is the saturation magnetization, $L(\zeta)$ is the Langevin function, m is the magnetic dipole moment, k_β is Boltzmann's constant, and T is temperature. The Brownian time of rotational diffusion τ_B can be calculated as $\tau_B = \frac{4\pi\alpha^3\mu}{k_\beta T}$. The physical meaning of the magnetization relaxation equation [Eq. (32)] is that the time change of the magnetization emerges from two main phenomena, i.e., the rotational motion of the suspended magnetic particles under the influence of an applied magnetic field and the tendency of the magnetic particles to attain the equilibrium magnetization. When no microrotation exists, i.e., when $\boldsymbol{\omega} = 0$, the magnetization attains its equilibrium value M_0 according to a classical exponential law, i.e., $\boldsymbol{M} - \boldsymbol{M_0} \sim e^{-t/\tau_B}$.

Using the definition of the magnetic susceptibility $\chi_m = \frac{M_0}{H}$,[17,25] the magnetization relaxation equation [Eq. (32)] can be written in steady state

$$\boldsymbol{M} = \frac{M_0}{H}[\boldsymbol{H} - \tau(\boldsymbol{H} \times \boldsymbol{\omega})], \tag{34}$$

Equation (34) represents the constitutive equation for magnetization as it is derived by Shizawa and Tanahashi.[25] In this equation, τ is the magnetization relaxation time, which is defined as $\tau = \tau_\beta(1 + \varepsilon)$, where $\varepsilon = \frac{\mu_r}{\mu}$ [see Eq. (73)]. In this study, this constitutive equation for the magnetization will be used to examine the effect of magnetization on the micropolar flow.

If the mother-liquid is conducting, then the full magnetohydrodynamic equations should be used in addition to the micropolar ferrofluid theory. This implies that the Lorentz force term should be added in the equation of conservation of linear momentum, while the full set of Maxwell's equations should be also utilized. In this context, the governing equations should be written as follows:

$$\rho\frac{D\boldsymbol{v}}{Dt} = -\nabla p + \mu\nabla^2\boldsymbol{v} + 2\mu_r\nabla\times(\boldsymbol{\omega}-\boldsymbol{\Omega}) + (\boldsymbol{M}\cdot\nabla)\boldsymbol{H} + \mu_0(\boldsymbol{j}\times\boldsymbol{H}), \tag{35}$$

$$\rho j\frac{D\boldsymbol{\omega}}{Dt} = 4\mu_r(\boldsymbol{\Omega}-\boldsymbol{\omega}) + \gamma\nabla^2\boldsymbol{\omega} + \boldsymbol{M}\times\boldsymbol{H}, \tag{36}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{37}$$

$$\nabla\times\boldsymbol{H} = \boldsymbol{j}, \tag{38}$$

$$\boldsymbol{j} = \sigma(\boldsymbol{v}\times\boldsymbol{B}). \tag{39}$$

C. Analytical solutions for conducting micropolar Couette and Poiseuille flow with magnetic particles

Consider a steady, laminar, 2D electrically conducting micropolar fluid with magnetic particles (for example, ferrofluid or blood) in two parallel infinite plates subjected to an applied magnetic field $\boldsymbol{H} = (0, H_0, 0)$, as shown in Fig. 1. The Cartesian coordinates $(\overline{x}, \overline{y}, \overline{z})$ are used in dimensional form. The flow is along the $\overline{x}$ axis, while the $\overline{y}$ axis is perpendicular to it. The origin of the axes is placed at the center plane of the plates located at $\overline{y} = \pm L$, where L is the half-distance between the plates. In the case of the Couette flow, the upper plane is moving at constant velocity V_0 with no pressure gradient in the direction of the flow. In the case of Poiseuille flow, no-slip conditions are applied for the velocity, while there is a pressure gradient $\frac{\partial\overline{p}}{\partial\overline{x}} = -G$ in the direction of the flow. Condiff–Dahler boundary conditions are used for microrotation and angular velocity for both flows [see Eqs. (62) and (63)]. The components of linear velocity, vorticity, and microrotation in dimensional form are given as $\boldsymbol{v} = (\overline{v}_x(\overline{y}), 0, 0)$, $\boldsymbol{\Omega} = \left(0, 0, \overline{\Omega}_z(\overline{y})\right)$, and $\boldsymbol{\omega} = (0, 0, \overline{\omega}_z(\overline{y}))$.

In this manner, the full magnetohydrodynamic equations for micropolar conducting fluids with magnetic particles [Eqs. (35) and (36) along with the definition of vorticity] for both Couette and Poiseuille flows are written in dimensional form:

$$\overline{\Omega}_z = -\frac{1}{2}\frac{\partial\overline{v}_x}{\partial\overline{y}}, \tag{40}$$

$$\frac{\partial\overline{p}}{\partial\overline{y}} = \mu_0\overline{j}_z\overline{H}_x, \tag{41}$$

$$\mu\frac{\partial^2\overline{v}_x}{\partial\overline{y}^2} + 2\mu_r\left(\frac{\partial\overline{\omega}_z}{\partial\overline{y}} - \frac{\partial\overline{\Omega}_z}{\partial\overline{y}}\right) - \mu_0\overline{j}_z\overline{H}_y + \overline{M}_y\frac{\partial\overline{H}_x}{\partial\overline{y}} = K, \tag{42}$$

$$\gamma\frac{\partial^2\overline{\omega}_z}{\partial\overline{y}^2} + 4\mu_r(\overline{\Omega}_z - \overline{\omega}_z) + \overline{M}_x\overline{H}_y - \overline{M}_y\overline{H}_x = 0, \tag{43}$$

where $K = 0$ for the case of the Couette flow and $K = -G$ for the case of the Poiseuille flow. Equation (40) represents the vorticity; Eqs. (41) and (42) are the conservation of linear momentum in $\overline{y}$ and $\overline{x}$ directions, respectively; and Eq. (43) is the change of internal angular momentum in $\overline{z}$ direction. Using Eq. (40), Eqs. (42) and (43) can be rewritten as

$$(\mu+\mu_r)\frac{\partial^2\overline{v}_x}{\partial\overline{y}^2} + 2\mu_r\frac{\partial\overline{\omega}_z}{\partial\overline{y}} - \mu_0\overline{j}_z\overline{H}_y + \overline{M}_y\frac{\partial\overline{H}_x}{\partial\overline{y}} = K, \tag{44}$$

$$\gamma\frac{\partial^2\overline{\omega}_z}{\partial\overline{y}^2} - 2\mu_r\frac{\partial\overline{v}_x}{\partial\overline{y}} - 4\mu_r\overline{\omega}_z + \overline{M}_x\overline{H}_y - \overline{M}_y\overline{H}_x = 0. \tag{45}$$

The form of Eqs. (44) and (45) is the most common one that is found in the literature.

Analyzing the dimensional form of the current density vector $\boldsymbol{j}$ [Eqs. (38) and (39)] in $\overline{x}$, $\overline{y}$, and $\overline{z}$ directions, it is obtained as follows:

$$\overline{j}_x = 0, \tag{46}$$

$$\overline{j}_y = 0, \tag{47}$$

$$\overline{j}_z = -\frac{\partial\overline{H}_x}{\partial\overline{y}} = \sigma\overline{v}_x\overline{B}_y. \tag{48}$$

Here, $\overline{H}_x$ is the induced magnetic field; hence, the dimensional form of the magnetic field vector is given as $\boldsymbol{H} = \left(\overline{H}_x, \overline{H}_y, 0\right)$. In this study, it is assumed that the induced magnetic field is smaller enough than the applied magnetic field to ignore its impact on the flow, i.e., $\frac{\overline{H}_x}{\overline{H}_y} \ll 1$. This assumption can be found in several studies regarding MHD flows,[10,27,28,45] by the name of "low magnetic Reynolds number approximation" $(Re_m \ll 1, Re_m = \sigma\mu_0 V_0 L)$. This approximation neglects the solution of the induction equation, which leads to the reduction of the equations that need to be solved. Thus, the magnetic field vector becomes $\boldsymbol{H} \cong \overline{H}_y$, which also implies that the magnitude of the magnetic field vector is $\overline{H} \cong \overline{H}_y$.

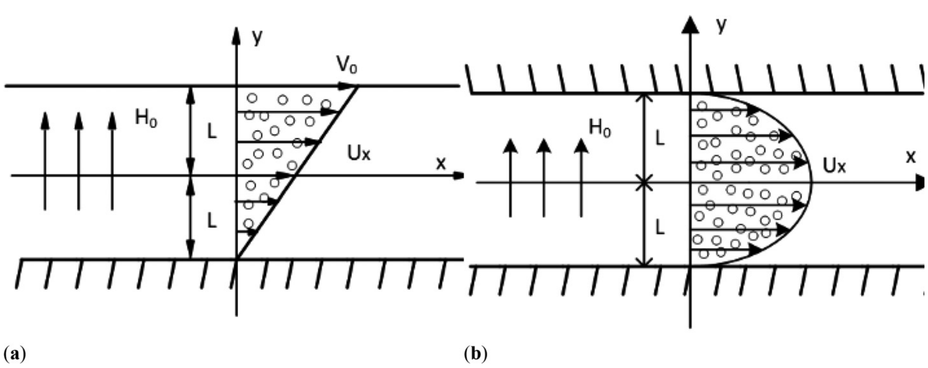

FIG. 1. Schematical representation of (a) conducting micropolar Couette flow with magnetic particles and (b) conducting micropolar Poiseuille flow with magnetic particles.

Using the low-Re_m approximation, the dimensional form of the constitutive equation for magnetization $\boldsymbol{M}$ Eq. 34) is analyzed as

$$\overline{M}_x = \frac{M_0}{\overline{H}}\left(\overline{H}_x - \tau\overline{H}_y\overline{\varpi}_z\right) \approx \frac{M_0}{\overline{H}}\left(\overline{H}_x - \tau\overline{H}\overline{\varpi}_z\right) = -\tau M_0\overline{\varpi}_z, \quad (49)$$

$$\overline{M}_y = \frac{M_0}{\overline{H}}\left(\overline{H}_y - \tau\overline{H}_x\overline{\varpi}_z\right) \approx \frac{M_0}{\overline{H}}\left(\overline{H} - \tau\overline{H}_x\overline{\varpi}_z\right) = M_0, \quad (50)$$

$$\overline{M}_z = 0. \quad (51)$$

In this manner, the magnetic induction vector $\boldsymbol{B}$ can be analyzed as

$$\overline{B}_x = \mu_0\overline{H}_x + \overline{M}_x \approx -\tau M_0\overline{\varpi}_z, \quad (52)$$

$$\overline{B}_y = \mu_0\overline{H}_y + \overline{M}_y \approx \mu_0\overline{H} + M_0, \quad (53)$$

$$\overline{B}_z = 0. \quad (54)$$

As it can be seen from Eq. (50), the magnetic fluid is permanently magnetized in the y direction with equilibrium magnetization M_0. On the other hand, the continuity of $\boldsymbol{B}$ across the plates requires that $\overline{B}_y = B_0 = \mu_0 H_0$. This leads to the derivation

$$\mu_0 H_0 = \mu_0\overline{H} + M_0 \quad (55)$$

or

$$\overline{H} = H_0 - \frac{M_0}{\mu_0}. \quad (56)$$

Equation (56) denotes the reduction of the total magnetic field inside the micropolar fluid.[10,16,28]

Finally, with the use of all the above assumptions, the governing equations in dimensional form are written as

$$\overline{\Omega}_z = -\frac{1}{2}\frac{\partial\overline{\upsilon}_x}{\partial\overline{y}}, \quad (57)$$

$$\mu\frac{\partial^2\overline{\upsilon}_x}{\partial\overline{y}^2} + 2\mu_r\left(\frac{\partial\overline{\varpi}_z}{\partial\overline{y}} - \frac{\partial\overline{\Omega}_z}{\partial\overline{y}}\right) - \sigma{\mu_0}^2{H_0}^2\overline{\upsilon}_x = K, \quad (58)$$

$$\gamma\frac{\partial^2\overline{\varpi}_z}{\partial\overline{y}^2} + 4\mu_r\left(\overline{\Omega}_z - \overline{\varpi}_z\right) - \tau H_0 M_0(1-h)\overline{\varpi}_z = 0, \quad (59)$$

or using Eq. (57) for $\overline{\Omega}_z$ in Eqs. (58) and (59),

$$(\mu + \mu_r)\frac{\partial^2\overline{\upsilon}_x}{\partial\overline{y}^2} + 2\mu_r\frac{\partial\overline{\varpi}_z}{\partial\overline{y}} - \sigma{\mu_0}^2{H_0}^2\overline{\upsilon}_x = K, \quad (60)$$

$$\gamma\frac{\partial^2\overline{\varpi}_z}{\partial\overline{y}^2} - 2\mu_r\frac{\partial\overline{\upsilon}_x}{\partial\overline{y}} - 4\mu_r\overline{\varpi}_z - \tau H_0 M_0(1-h)\overline{\varpi}_z = 0, \quad (61)$$

where $h = \frac{M_0}{\mu_0 H_0}$. As it is mentioned above, no-slip boundary conditions are used for velocity, while Condiff–Dahler boundary conditions are used for microrotation and angular velocity as follows:

Couette flow

$$\overline{\upsilon}_x(-L) = 0, \;\; \overline{\upsilon}_x(L) = V_0, \;\; \overline{\varpi}_z(-L) = \delta\overline{\Omega}_z(-L), \;\; \overline{\varpi}_z(L) = \delta\overline{\Omega}_z(L). \quad (62)$$

Poiseuille flow

$$\overline{\upsilon}_x(-L) = 0, \;\; \overline{\upsilon}_x(L) = 0, \;\; \overline{\varpi}_z(-L) = \delta\overline{\Omega}_z(-L), \;\; \overline{\varpi}_z(L) = \delta\overline{\Omega}_z(L). \quad (63)$$

The parameter δ is the microrotation wall coefficient and its effect on the flow will be discussed in Sec. III F.

By differentiating the equation of conservation of linear momentum [Eq. (60)] and using the equation of change of the internal angular momentum [Eq. (61)], a one-way coupled differential equation system is derived as follows:

$$\overline{\varpi}_z = \overline{F}\frac{\partial^3\overline{\upsilon}_x}{\partial\overline{y}^3} + \overline{Z}\frac{\partial\overline{\upsilon}_x}{\partial\overline{y}}, \quad (64)$$

$$\overline{A}\frac{\partial^4\overline{\upsilon}_x}{\partial\overline{y}^4} + \overline{B}\frac{\partial^2\overline{\upsilon}_x}{\partial\overline{y}^2} + \overline{\Gamma}\overline{\upsilon}_x - K = 0, \quad (65)$$

where

$$\overline{F} = -\frac{\gamma(\mu + \mu_r)}{2\mu_r\left[4\mu_r + \tau H_0 M_0(1-h)\right]}, \quad (66)$$

$$\overline{Z} = \frac{\gamma\sigma{\mu_0}^2{H_0}^2}{2\mu_r\left[4\mu_r + \tau H_0 M_0(1-h)\right]} - \frac{2\mu_r}{4\mu_r + \tau H_0 M_0(1-h)}, \quad (67)$$

$$\overline{A} = 2\mu_r F, \quad (68)$$

$$\overline{B} = \mu + \mu_r + 2\mu_r Z, \quad (69)$$

$$\overline{\Gamma} = -\sigma{\mu_0}^2{H_0}^2. \quad (70)$$

In order for the governing equations to be written in non-dimensional form, the following dimensionless variables are used:

Couette flow

$$y = \frac{\overline{y}}{L}, \quad \upsilon_x = \frac{\overline{\upsilon}_x}{V_0}, \quad \Omega_z = \overline{\Omega}_z\frac{L}{V_0}, \quad \omega_z = \overline{\varpi}_z\frac{L}{V_0}, \quad (71)$$

Poiseuille flow

$$y = \frac{\overline{y}}{L}, \quad \upsilon_x = \overline{\upsilon}_x\frac{\mu}{GL^2}, \quad \Omega_z = \overline{\Omega}_z\frac{\mu}{GL}, \quad \omega_z = \overline{\varpi}_z\frac{\mu}{GL}. \quad (72)$$

In this manner, the following dimensionless parameters are introduced:

$$\varepsilon = \frac{\mu_r}{\mu}, \quad \lambda = \frac{L}{\imath}, \quad Ha = \mu_0 H_0 L\sqrt{\frac{\sigma}{\mu}}, \quad \sigma_m = \frac{\tau\tau_s H_0 M_0}{I}(1-h), \quad (73)$$

where $\imath = \sqrt{j}$. In Eq. (73), ε is the micropolar effect parameter, λ is the size effect parameter, Ha is the Hartmann number, and σ_m is the magnetization effect parameter. All these parameters will be discussed in Sec. III A. It should be mentioned that in the papers of Shliomis and Tanahashi,[25,42,43] it is derived that $\varepsilon = \frac{3}{2}\varphi$.

Using the variables introduced in Eqs. (71) and (72) and the dimensionless parameters in Eq. (73), Eqs. (64) and (65) along with the vorticity [Eq. (57)] are reformed as follows:

$$\Omega_z = -\frac{1}{2}\frac{\partial\upsilon_x}{\partial y}, \quad (74)$$

$$\omega_z = F\frac{\partial^3\upsilon_x}{\partial y^3} + Z\frac{\partial\upsilon_x}{\partial y}, \quad (75)$$

$$\frac{\partial^4\upsilon_x}{\partial y^4} + A\frac{\partial^2\upsilon_x}{\partial y^2} + B\upsilon_x - K = 0, \quad (76)$$

where $K = 0$ for the Couette flow, $K = \frac{4\varepsilon\lambda^2(1+\sigma_m)}{1+\varepsilon}$ for the Poiseuille flow, and

$$F = -\frac{1+\varepsilon}{8\varepsilon^2\lambda^2(1+\sigma_m)}, \tag{77}$$

$$Z = \frac{Ha^2 - 4\varepsilon^2\lambda^2}{8\varepsilon^2\lambda^2(1+\sigma_m)}, \tag{78}$$

$$A = -\left[4\varepsilon\lambda^2\sigma_m + \frac{Ha^2 + 4\varepsilon\lambda^2}{1+\varepsilon}\right], \tag{79}$$

$$B = \frac{4\varepsilon\lambda^2(1+\sigma_m)Ha^2}{1+\varepsilon}. \tag{80}$$

The boundary conditions in non-dimensional form for both velocity and microrotation are written as follows:

Couette flow

$$\upsilon_x(-1) = 0, \quad \upsilon_x(1) = 1, \quad \omega_z(-1) = \delta\Omega_z(-1), \quad \omega_z(1) = \delta\Omega_z(1), \tag{81}$$

Poiseuille flow

$$\upsilon_x(-1) = 0, \quad \upsilon_x(1) = 0, \quad \omega_z(-1) = \delta\Omega_z(-1), \quad \omega_z(1) = \delta\Omega_z(1). \tag{82}$$

The solutions of the differential equation system [Eqs. (74)–(76)] are calculated as follows:

$$\upsilon_x = C4e^{-Ax} + C3e^{Ax} + C2e^{-Bx} + C1e^{Bx} + \frac{\xi 3}{\xi 2}, \tag{83}$$

$$\omega_z = e^{-(A+B)x}\left(-Be^{Ax}(C2 - C1e^{2Bx})K + B^3e^{Ax}(C2 - C1e^{2Bx})\Lambda + Ae^{Bx}(C4 - C3e^{2Ax})(-K + A^2\Lambda)\right). \tag{84}$$

The analytical solutions for both velocity and microrotation were derived with the use of the Wolfram Mathematica software. All variables appearing in Eqs. (83) and (84) can be found in the Appendix.

One can notice that the system of the differential equation system for conducting Couette and Poiseuille micropolar flows with magnetic particles can be reduced to the corresponding simple micropolar flows when certain dimensionless parameters take zero value, i.e., for $Ha = 0$ and $\sigma_m = 0$. In this manner, the differential equation system in Eqs. (74)–(76) stays the same, but its parameters are reformed as follows:

$$F = -\frac{1+\varepsilon}{8\varepsilon^2\lambda^2}, \tag{85}$$

$$Z = -\frac{1}{2}, \tag{86}$$

$$A = -\frac{4\varepsilon\lambda^2}{1+\varepsilon}, \tag{87}$$

$$B = 0, \tag{88}$$

while $K = 0$ for the Couette flow and $K = \frac{4\varepsilon\lambda^2}{1+\varepsilon}$ for the Poiseuille flow.

Moreover, the differential equation system in Eqs. (74)–(76) can be reduced to solve MHD micropolar Poiseuille and Couette flows without magnetic particles when $\sigma_m = 0$. Such flows have been studied by various researchers, while this model has been used for the study of micropolar blood flows under the influence of externally applied magnetic fields (i.e., MRI cases), where the magnetization of the red blood cells is ignored.[9,30,46,47] As in the case of the simple micropolar flows, the differential equation system in Eqs. (74)–(76) stays the same, but its parameters are reformed as follows:

$$F = -\frac{1+\varepsilon}{8\varepsilon^2\lambda^2}, \tag{89}$$

$$Z = \frac{Ha^2}{8\varepsilon^2\lambda^2} - \frac{1}{2}, \tag{90}$$

$$A = -\frac{Ha^2 + 4\varepsilon\lambda^2}{1+\varepsilon}, \tag{91}$$

$$B = \frac{4\varepsilon\lambda^2 Ha^2}{1+\varepsilon}, \tag{92}$$

while K is the same as in the case of the simple micropolar Couette and Poiseuille flows.

III. ANALYTICAL RESULTS AND DISCUSSION

A. Discussion of the dimensionless parameters

The main objective of this paper is to examine the mechanisms found in micropolar MHD fluids with magnetic particles that impose or prevent the microrotation field (i.e., the internal rotation of the suspended particles in the fluid) to be equal to fluid's vorticity, i.e., $\boldsymbol{\omega} = \boldsymbol{\Omega}$. For this study, two types of micropolar MHD fluids with magnetic particles are utilized, blood and ferrofluid, which are found in a variety of biomedical and engineering applications. In order to continue this examination, the dimensionless parameters associated with such flows [see Eq. (73)] should be analyzed first.

One of the most important dimensionless parameters associated with micropolar fluids is the micropolar effect parameter ε. This parameter is a measure of the micropolarity of the fluid. According to Eq. (73), ε is proportional to the rotational viscosity μ_r, which means that as μ_r increases, so does ε. This can happen in two cases; either the sum of the particles' moment of inertia per unit volume I increases or the spin relaxation time τ_s decreases [see Eq. (14)]. On the other hand, as can be seen from Eq. (73), ε is inversely proportional to the dynamical viscosity of the fluid μ. Finally, as it is mentioned in Sec. II C, $\varepsilon = \frac{3}{2}\varphi$, which indicates that when φ increases, ε increases too.

Another important parameter for the micropolar fluids is the size effect parameter λ. According to Eq. (73), λ is proportional to the channel's half-height L and inversely proportional to the square root of microinertia coefficient j. Through j, the size effect parameter is also inversely proportional to the spin viscosity γ [see Eq. (23)]. This means that the size effect parameter decreases as the microrotation's diffusion increases. Moreover, λ is associated with the role of the of the couple stress tensor $\boldsymbol{C}$ on the micropolar flow, which is ignored in many ferrofluid studies.[16–18,24,26,46,48]

A third parameter found in this study is the Hartmann number Ha. It is a commonly found parameter in MHD studies.[49–51] Ha is associated with higher values of the applied magnetic field H_0 or/and higher values of the electrical conductivity σ. It is obvious from Eq. (73) that $Ha = 0$ when no external applied magnetic field is applied on the flow ($H_0 = 0$) and/or when the fluid is non-conducting ($\sigma = 0$).

Finally, the magnetization effect parameter σ_m is another dimensionless parameter found in this study. It is associated with the effect of micromagnetorotation (MMR) on the micropolar flow (see Aslani *et al.*[10,27,28]). One can easily see from Eq. (73) that $\sigma_m = 0$ when no external applied magnetic field is applied on the flow ($H_0 = 0$) and/or when the micropolar fluid does not contain magnetic particles ($M_0 = 0$ and $\tau = 0$).

B. Fluids' parameters

As it is mentioned above, in this study, two types of micropolar fluids are utilized for the examination of the mechanisms, which affect the difference $\boldsymbol{\omega} - \boldsymbol{\Omega}$, namely, blood and ferrofluid. Both fluids are found in a variety of biomedical and engineering applications.[52,53] Moreover, they contain particles with magnetic properties, which experience magnetic torque when an external magnetic field is applied. Here, the cases where these fluids can be modeled as Newtonian ones will be identified, which can lead to the simplification of the governing equations system and the reduction of the computational cost (both time and resources).

First, the fluids' parameters should be derived using the physical properties of the ferrofluid and blood. Here, the experimental ferrofluid, namely, EMG-206 from Ferrotec is used, which has been also used by other studies considering ferrofluid flows.[23] Table I shows ferrofluid's and blood's physical properties as they were derived from various relevant studies.[22,23,34]

It should be mentioned that the volume fraction φ of the blood coincides with the hematocrit, which in this study is set to the typical value of 0.45. Moreover, the height of the channel flow $2L$ has been set to 10^{-3} m, which is a typical value for arterioles.[22] The value of blood's electrical conductivity as it can be seen in Table I is taken without the consideration of the electrolyte content, including salts, colloidal electrolytes, and proteins; this value has been used in a variety of numerical and experimental studies; other investigations consider a typical value of $\sigma = 0.6\ \frac{S}{m}$ for blood, which is 10 times higher due to the electrolytes' concentration.[35,54] The absolute temperature of $309.75\,K$ ($36.6\,^\circ$C) has been chosen, which is the temperature of the human body. Both channel's height and absolute temperature are also used in the case of the ferrofluid for the sake of comparison with the blood. Using all relevant mathematical relations for the calculation of the fluids' parameters (see Sec. II), the latter can be seen in Table II.

For the examination of the impact of the Lorentz force and the micromagnetorotation term on the microrotation–vorticity difference $\boldsymbol{\omega} - \boldsymbol{\Omega}$, three values of the applied magnetic field's intensity are chosen,

TABLE I. Physical properties of ferrofluid and blood.

	Ferrofluid (EMG-206)	Blood
Volume fraction φ (%)	33.33	45
Dynamical viscosity μ (Pa s)	3.85×10^{-3}	4×10^{-3}
Fluid density ρ_f (kg m^{-3})	1 187.4	1 050
Particles' radius α (m)	5×10^{-9}	4×10^{-6}
Particle density ρ_α (kg m^{-3})	5 420	1 100
Saturation magnetization M_s (A m^{-1})	11 940	100
Electrical conductivity σ (S m^{-1})	0.5	0.067

TABLE II. Calculated ferrofluid and blood parameters.

	Ferrofluid (EMG-206)	Blood
Micropolar effect parameter ε (-)	0.5	0.675
Rotational viscosity μ_r (Pas)	1.925×10^{-3}	2.7×10^{-3}
Microinertia coefficient j (m^2)	6.086×10^{-17}	7.5×10^{-10}
Spin relaxation time τ_s (s)	9.39×10^{-12}	7.29×10^{-5}
Spin viscosity γ ($\frac{\text{kgm}}{\text{s}}$)	2.34×10^{-15}	3×10^{-13}
Size effect parameter λ (-)	640.95	18.26
Magnetization relaxation time τ (s)	11×10^{-6}	10^{-3}

TABLE III. Calculated values for the Hartmann number Ha and the magnetization effect parameter σ_m.

	Magnetic field intensity	Ferrofluid (EMG-206)	Blood
Ha values	$H_0 = 795\,774.72\,\frac{\text{A}}{\text{m}}$ (1 T)	0.0057	0.002
	$H_0 = 2\,387\,324.15\,\frac{\text{A}}{\text{m}}$ (3 T)	0.017	0.0061
	$H_0 = 6\,366\,197.72\,\frac{\text{A}}{\text{m}}$ (8 T)	0.046	0.016
σ_m values	$H_0 = 795\,774.72\,\frac{\text{A}}{\text{m}}$ (1 T)	5.79	9.26
	$H_0 = 2\,387\,324.15\,\frac{\text{A}}{\text{m}}$ (3 T)	17.48	27.78
	$H_0 = 6\,366\,197.72\,\frac{\text{A}}{\text{m}}$ (8 T)	46.65	74.1

which are frequently found in experimental and numerical studies associated with biomedical and engineering applications, i.e., $H_0 = 795\,774.72\,\frac{\text{A}}{\text{m}}$, $H_0 = 2\,387\,324.15\,\frac{\text{A}}{\text{m}}$, and $H_0 = 6\,366\,197.72\,\frac{\text{A}}{\text{m}}$.[35,55,56] Reminding that $H_0 = \frac{B_0}{\mu_0}$, which leads to $H_0 = 795\,774.72\,\frac{\text{A}}{\text{m}}$ corresponding to $1\,T$, $H_0 = 2\,387\,324.15\,\frac{\text{A}}{\text{m}}$ corresponding to $3\,T$, and $H_0 = 6\,366\,197.72\,\frac{\text{A}}{\text{m}}$ corresponding to 8 T. In this manner, Table III shows the values of the Hartmann number Ha and the magnetization effect parameter σ_m for all the considered values of the applied magnetic field both for the ferrofluid and the blood.

Finally, using the values for the electrical conductivity of both blood and ferrofluid (see Table I) and considering that $\mu_0 = 4\pi \times 10^{-7}\,\frac{\text{H}}{\text{m}}$ and $2L = 10^{-3}$ m, the magnetic Reynolds number $Re_m = \sigma\mu_0 V_0 L$ is calculated as $Re_m = 3.14 \times 10^{-10}$ for the ferrofluid and $Re_m = 4.2 \times 10^{-11}$ for blood. This means that the "low magnetic Reynolds number approximation" (see Sec. II C) holds and the induced magnetic field can be ignored.

C. Microrotation–vorticity difference for simple micropolar Couette and Poiseuille flows

In this paragraph, the case of $\boldsymbol{\omega} = \boldsymbol{\Omega}$ for simple 2D micropolar blood and ferrofluid flows (Couette and Poiseuille). This means that the terms of the Lorentz force and the micromagnetorotation in the governing differential equation system [Eqs. (57)–(59)] are ignored. As it is mentioned in the previous section, this corresponds to the case of $Ha = 0$ and $\sigma_m = 0$. For blood and ferrofluid flows, it means that no external magnetic field is applied ($H_0 = 0$). The final differential equation system that describes such flows constitutes of Eqs. (74)–(76) and the parameters in Eqs. (85) and (88).

In general, when $\boldsymbol{\omega} = \boldsymbol{\Omega}$ in 2D micropolar Couette and Poiseuille flows, it is implied that Eq. (75) should be equal to Eq. (74) i.e., $F\frac{\partial^3 v_x}{\partial y^3} + Z\frac{\partial v_x}{\partial y} = -\frac{1}{2}\frac{\partial v_x}{\partial y}$. Considering that $Z = -\frac{1}{2}$ [Eq. (86)], then $F = -\frac{1+\varepsilon}{8\varepsilon^2\lambda^2}$ [Eq. (85)] should equal to zero. One can easily see that F is a function of the micropolar effect parameter ε and the size effect parameter λ. In this manner, when $\varepsilon = 0$ and/or $\lambda = 0$, F takes an indeterminate form. Is it possible that $F \to 0$ without taking the indeterminate form that is accompanied by the cases of $\varepsilon = 0$ and/or $\lambda = 0$?

Figure 2 shows how F changes as ε increases for a specific value of λ, which corresponds to ferrofluid (left side) and blood (right side). It is evident that F tends to zero for higher values of ε for both fluids. This means that for higher values of the rotational viscosity μ_r (given that the dynamical viscosity μ does not change), then $\omega_z = \Omega_z$. This a contradiction to the common belief that higher values of μ_r (and in turn, higher values of ε) lead to higher micropolarity. As mentioned to Sec. III B, μ_r increases either for higher values of I or for lower values of τ_s. Given that when I increases, $\imath$ increases too (see Sec. II) and λ decreases, then someone can conclude that the reduction of the spin relaxation time τ_s leads to $\omega_z \to \Omega_z$.

As mentioned in the Introduction, this spin relaxation time phenomenon has been described by Shliomis in his work with ferrofluids.[14,17,18,24] One main problem with this type of fluids is that in the absence of an external magnetic field, the suspended particles are only influenced by the friction forces exerted by the mother liquid, which are not enough to maintain a difference between $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$. In this case, the spin relaxation time is very small, which means that the equalization of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ happens very quickly and any differences between the latter are not noticeable. Then, the flow is described by the classical model of the Navier–Stokes equations. Given that in our study $\varepsilon = 0.5$ for the ferrofluid, from Fig. 2(a), Shliomis' assumption seems to be valid.

It is also obvious from Fig. 2 that λ also affects greatly the value of F. In the case of $\lambda = 640.95$ (ferrofluid), $F \leq 7.5 \times 10^{-4}$ (in absolute value), which is close to zero. On the other hand, in the case of $\lambda = 18.26$ (blood), it is given that $F \leq 0.9$ (in absolute value), which is not that close to zero. Specifically, for the ferrofluid ($\varepsilon = 0.5$), it is given that $F = -1.83 \times 10^{-6}$, which is practically zero, whereas for blood ($\varepsilon = 0.675$), it is given that $F = -0.0014$, a value that should not be ignored. This means that higher values of λ lead to $F \to 0$. This is easy to explain; as λ increases (given that L does not change), the influence of γ through j decreases, which leads to smaller microrotation diffusion and the couple stress tensor $\boldsymbol{C}$ can be ignored (a common practice that is found in ferrofluid studies[16–18,24,26]). In this case, the micropolar fluid equations are reduced to the ones of a Newtonian fluid. On the other hand, as the channel's height increases, λ also increases. This indicates that the size of the channel is an important parameter in micropolar fluids and should be always checked before someone ignores the microrotation's diffusion. For example, considering our results, the consideration of blood as a Newtonian fluid is not valid for channels with height $\lesssim 10^{-2}$ m, which is the case of arteries and arterioles that is used here.

Figure 3 shows the velocity v_x, the vorticity Ω_z, the microrotation ω_z, and the microrotation–vorticity difference $\omega_z - \Omega_z$ for simple micropolar ferrofluid and blood flows (Couette flow at the left and Poiseuille flow at the right). As expected, the velocity and the vorticity of the ferrofluid is very close to the Newtonian case, which validates Shliomis' assumption that ferrofluids can be considered Newtonian fluids when no magnetic field is applied. Microrotation coincides with vorticity leading to zero $\omega_z - \Omega_z$ everywhere in the channel, a situation that was expected due to high λ values and small spin relaxation time τ_s.

On the other hand, blood's micropolar nature is more evident. Its velocity and vorticity differ from the Newtonian one, even when no magnetic field is applied. Microrotation differs from vorticity leading to $\omega_z - \Omega_z \neq 0$, especially near channel's walls, where micropolar effects are more intense. Following previous discussion about blood's value for F, it is concluded that blood should be modeled as a micropolar fluid, especially when the height of the channel is small, a valid case for human vessels.

D. Microrotation–vorticity difference for MHD micropolar Couette and Poiseuille flows without magnetic particles

In this section, the microrotation–vorticity difference $\boldsymbol{\omega} - \boldsymbol{\Omega}$ is examined for 2D MHD micropolar Couette and Poiseuille flows without magnetic particles (i.e., only the effect of the Lorentz force was taken into account). Here, only blood was used due to the existence of numerous studies that do not consider erythrocytes as magnetic

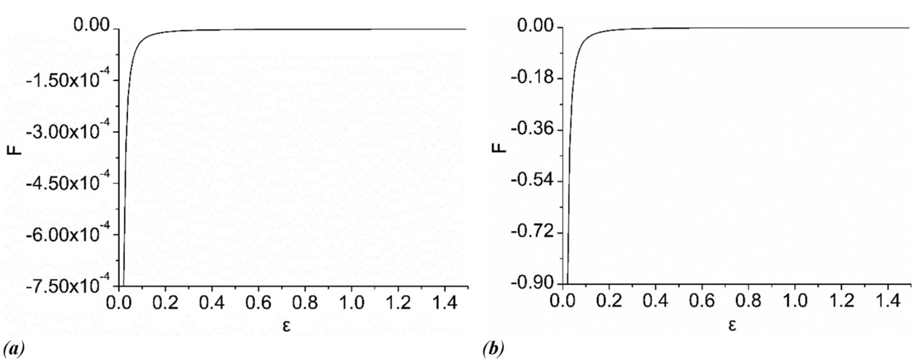

FIG. 2. Plot of $F(\varepsilon)$ for (a) ferrofluid and (b) blood (simple micropolar Couette and Poiseuille flows).

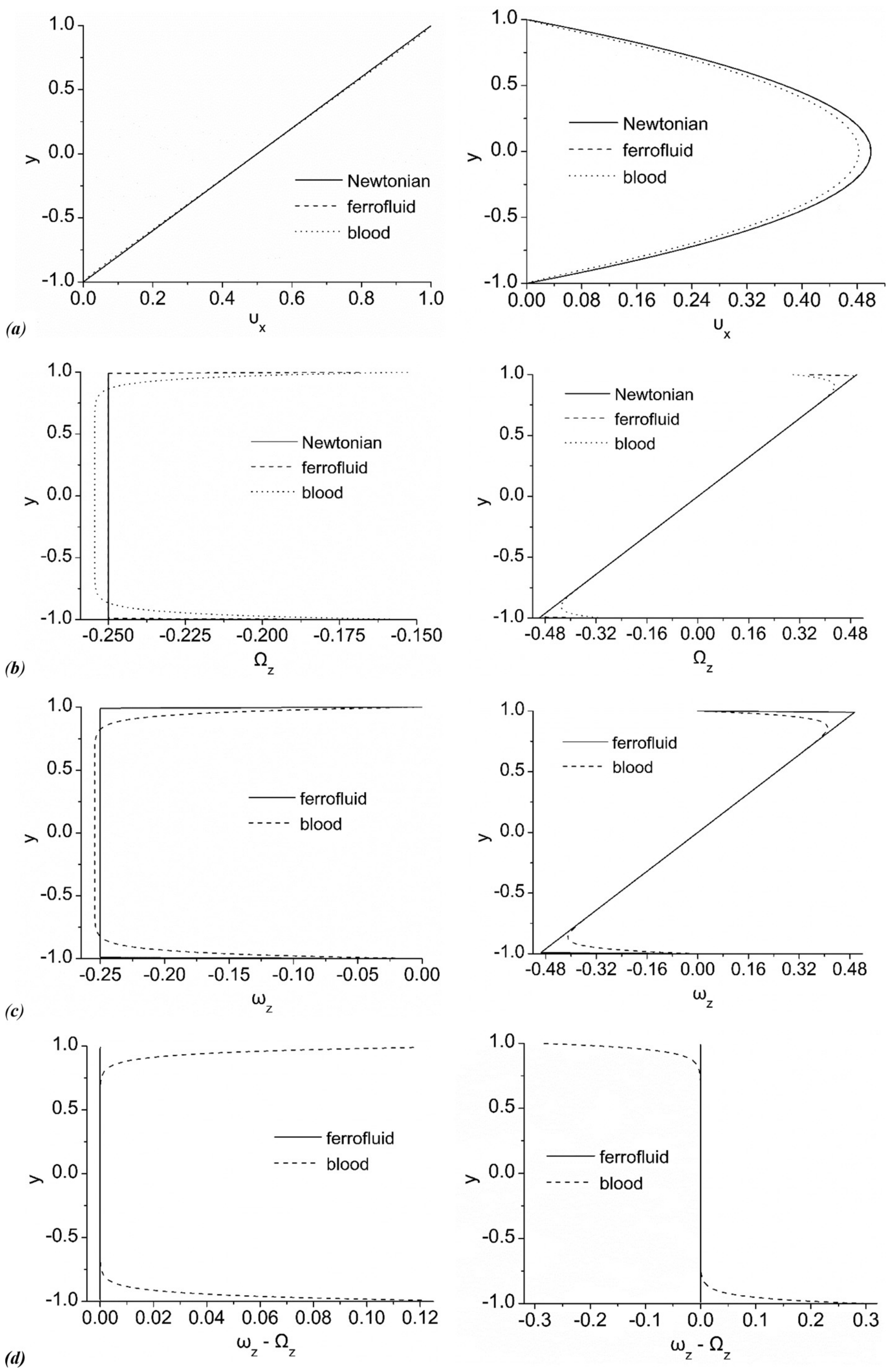

FIG. 3. Plots of (a) Velocity υ_x, (b) Vorticity Ω_z, (c) Microrotation ω_z, and (d) Microrotation–Vorticity difference $\omega_z - \Omega_z$ for simple micropolar Couette (left) and Poiseuille (right) flows (ferrofluid and blood are considered).

particles. Ferrofluids are characterized by the magnetization of their particles; there is no point of an examination of a ferrofluid as an MHD micropolar fluid. For blood, three different intensities of the applied magnetic field were used which are shown in Table III. In order to specify if the externally applied magnetic field affects $\boldsymbol{\omega} - \boldsymbol{\Omega}$ when fluid's particles are not magnetic, the governing differential equation system of Eqs. (74)–(76) and the parameters in Eqs. (89)–(92) are used. As in the case of the simple micropolar flows, in order for $\omega_z = \Omega_z$, Eq. (75) should be equal to Eq. (74), i.e., $F\frac{\partial^3 \upsilon_x}{\partial y^3} + Z\frac{\partial \upsilon_x}{\partial y} = -\frac{1}{2}\frac{\partial \upsilon_x}{\partial y}$. Here, $F = -\frac{1+\varepsilon}{8\varepsilon^2\lambda^2}$ [Eq. (89)], which is the same

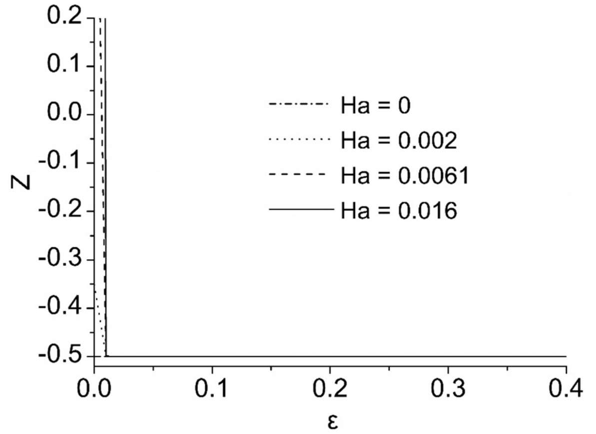

FIG. 4. Plot of $Z(\varepsilon)$ of blood for different Hartmann number values (MHD micropolar Couette and Poiseuille flows).

with the simple micropolar case, and it is depicted in Fig. 2(b). Moreover, $Z = \frac{Ha^2}{8\varepsilon^2\lambda^2} - \frac{1}{2}$ [Eq. (90)], which should be equal to $-\frac{1}{2}$.

Figure 4 shows how Z changes with respect to the micropolar effect parameter ε for different Hartmann values $H\alpha$ (given that λ does not change). It is evident that when only the effect of the Lorentz force is considered, the application of a magnetic field on a blood flow does not affect its micropolarity. $Z \cong -\frac{1}{2}$ almost for all ε values, for all Hartmann numbers. Given that F remains the same as with the simple micropolar Couette and Poiseuille flows, a blood flow under the influence of strong magnetic fields has the same micropolarity as when no magnetic field is applied.

Figure 5 shows the velocity υ_x, the vorticity Ω_z, the microrotation ω_z, and the microrotation–vorticity difference $\omega_z - \Omega_z$ for MHD micropolar Couette and Poiseuille blood flows using different Hartmann numbers. As expected from the above discussion, the magnetic field does not seem to have any effect on the blood flow. Given that the magnetization of the erythrocytes is not considered and the influence of the Lorentz force is not strong enough to affect blood's velocity (blood has small electrical conductivity), the magnetic field does not alternate neither the microrotation nor the microrotation–vorticity difference.

E. Microrotation–vorticity difference for MHD micropolar Couette and Poiseuille flows with magnetic particles

In this this section, the microrotation–vorticity difference $\boldsymbol{\omega} - \boldsymbol{\Omega}$ is examined for 2D MHD micropolar Couette and Poiseuille flows with magnetic particles (i.e., both the effect of the Lorentz force and the micromagnetorotation are taken into account). Here, both ferrofluid and blood are utilized. Again, the same intensities of the applied magnetic field as in Sec. III D are used (which are shown in Table III along with the corresponding values for the Hartmann number and the microrotation effect parameter). In order to specify how the Lorentz force combined with particles' magnetic torque affect $\boldsymbol{\omega} - \boldsymbol{\Omega}$, the governing differential equation system of Eqs. (74)–(76) and the parameters in Eqs. (77)–(80) are used. As in the previous two cases, in order for $\omega_z = \Omega_z$, Eq. (75) should be equal to Eq. (74), i.e., $F\frac{\partial^3 \upsilon_x}{\partial y^3} + Z\frac{\partial \upsilon_x}{\partial y} = -\frac{1}{2}\frac{\partial \upsilon_x}{\partial y}$. Here, $F = -\frac{1+\varepsilon}{8\varepsilon^2\lambda^2(1+\sigma_m)}$ (Eq. 77), which should be equal to zero and $Z = \frac{Ha^2 - 4\varepsilon^2\lambda^2}{8\varepsilon^2\lambda^2(1+\sigma_m)}$ [Eq. (78)], which should be equal to $-\frac{1}{2}$.

Figure 6 shows how F and Z change with respect to the micropolar effect parameter ε for different Hartmann numbers $H\alpha$ and microrotation effect parameter σ_m values. At the left side, the ferrofluid is depicted, while blood is depicted at the right side. The size effect parameter λ does not change; it is set to the corresponding values for every fluid. It is evident that when the Lorentz force combined with micromagnetorotation are considered, the behavior of F does not change dramatically compared with the simple micropolar Couette and Poiseuille cases. F goes to zero quickly, while all F values are a little smaller compared with the cases of $Ha = 0$ and $\sigma_m = 0$.

On the other hand, the results for Z are different. In the case of the ferrofluid, Z does not tend to $-\frac{1}{2}$ for all ε values, for all σ_m values. Similarly, in the case of blood, $Z \to -\frac{1}{2}$ only for $\sigma_m = 9.26$ (which corresponds to $Ha = 0.002$ and a magnetic field of 1 T, see Table III). For the other two σ_m values, Z does not tend to $-\frac{1}{2}$ for all ε values. It seems that when micromagnetorotation is considered, the micropolar nature of a fluid with magnetic particles is strengthened, as it was claimed by Shliomis and others.[16–18,24,26] Microrotation is differentiated from vorticity through Z, whereas F does not seem to alter the result for all ε values. Moreover, it is also proved that blood's micropolarity is affected by the application of an external magnetic field due to the magnetization of the erythrocytes. Ferrofluids and blood do not behave as Newtonian fluids when a magnetic field is applied as the antisymmetric part of the stress tensor should be taken into account due to the existence of magnetic particles in the flow.

Figure 7 shows the velocity υ_x, the vorticity Ω_z, the microrotation ω_z, and the microrotation–vorticity difference $\omega_z - \Omega_z$ for MHD micropolar Couette and Poiseuille ferrofluid flows using different Hartmann numbers and microrotation effect parameter values, which correspond to different intensities of the applied magnetic field (see Table III). As expected, the magnetic field affects the microrotation ω_z of the ferrofluid, which is reduced as σ_m increases, while it is also differentiated from vorticity Ω_z, resulting in $\omega_z - \Omega_z \neq 0$ for both flow types. Vorticity slightly increases as σ_m increases, whereas velocity slightly decreases. These are typical results for the behavior of a ferrofluid under the influence of an externally applied magnetic field, where the torque of the magnetic particles is taken into account. It is a standard practice that ferrofluids should not be considered as Newtonian fluids when a magnetic field is applied.

Figure 8 shows the velocity υ_x, the vorticity Ω_z, the microrotation ω_z, and the microrotation–vorticity difference $\omega_z - \Omega_z$ for MHD micropolar Couette and Poiseuille blood flows. As in the case of the ferrofluid, the magnetic field affects greatly erythrocytes' microrotation ω_z, which is reduced as σ_m increases, i.e., as the intensity of the applied magnetic field increases. Moreover, microrotation is differentiated from vorticity Ω_z, resulting in $\omega_z - \Omega_z \neq 0$ for both flow types, a situation that proves blood's micropolarity. Here, both vorticity and velocity slightly decrease as σ_m increases. These results prove that an externally applied magnetic field on a blood flow affects erythrocytes' microrotation due to their magnetization, which in turn affects blood's velocity through vorticity, a phenomenon that is mentioned in many experimental studies.[22,55–58] It seems that blood should not be modeled as a Newtonian fluid and its micropolarity should be taken into account, especially in the case of an applied magnetic field.

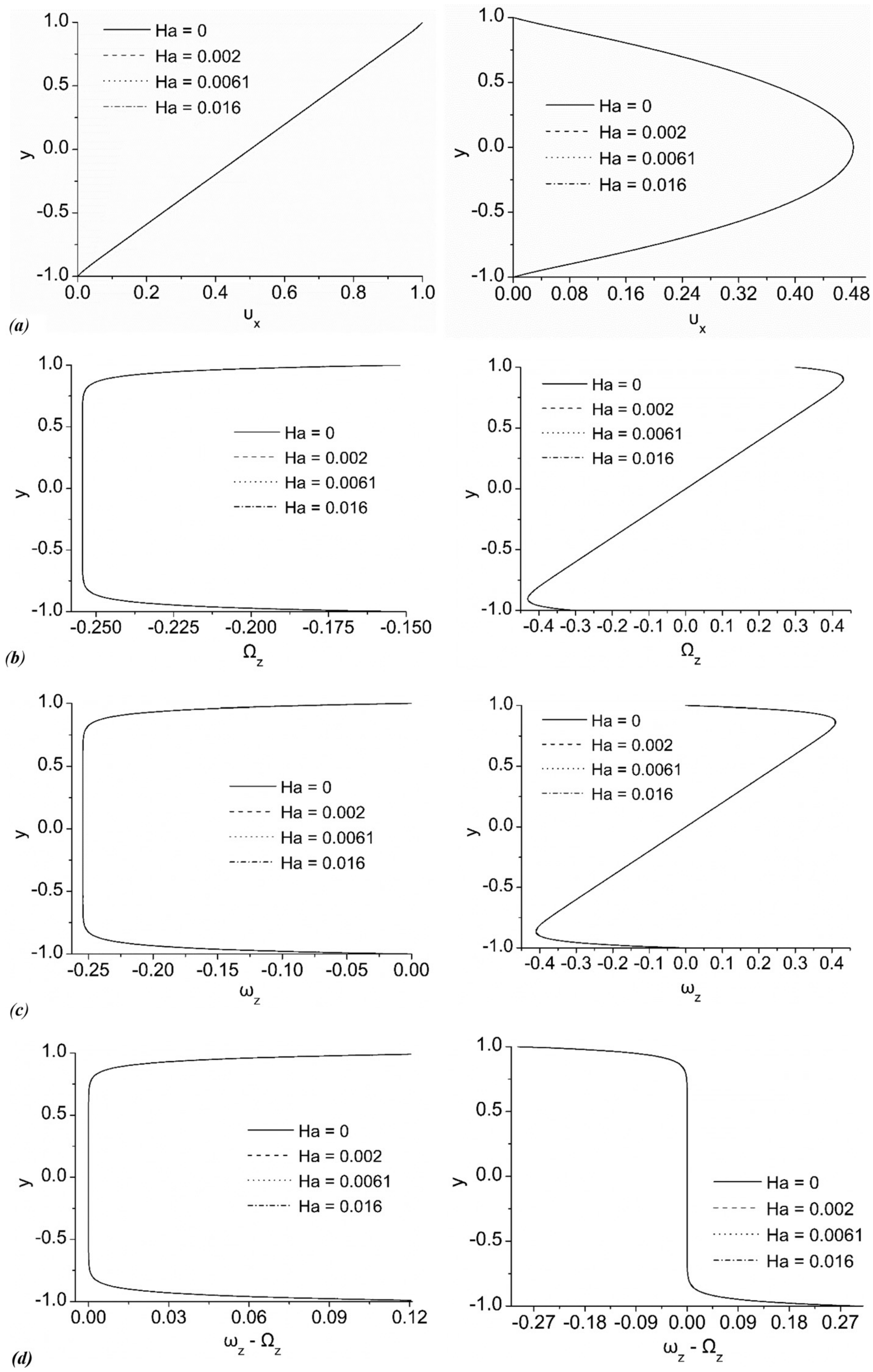

FIG. 5. Plots of (a) Blood's velocity υ_x, (b) Vorticity Ω_z, (c) Microrotation ω_z, and (d) Microrotation–Vorticity difference $\omega_z - \Omega_z$ for MHD micropolar Couette (left) and Poiseuille (right) flows using different Hartmann number values.

F. Effect of microrotation boundary conditions on microrotation–vorticity difference

One of the most important issues in micropolar fluids is the microrotation boundary conditions.[59–61] As mentioned by Condiff and Dahler,[2] the no-slip condition imposed on velocity does not allow the suspended particles to move relative to channel's walls, i.e., $\boldsymbol{\omega} = 0$ at the walls. The forces that originate on the wall dominate the forces due to particle interactions. It is reasonable that these wall forces that

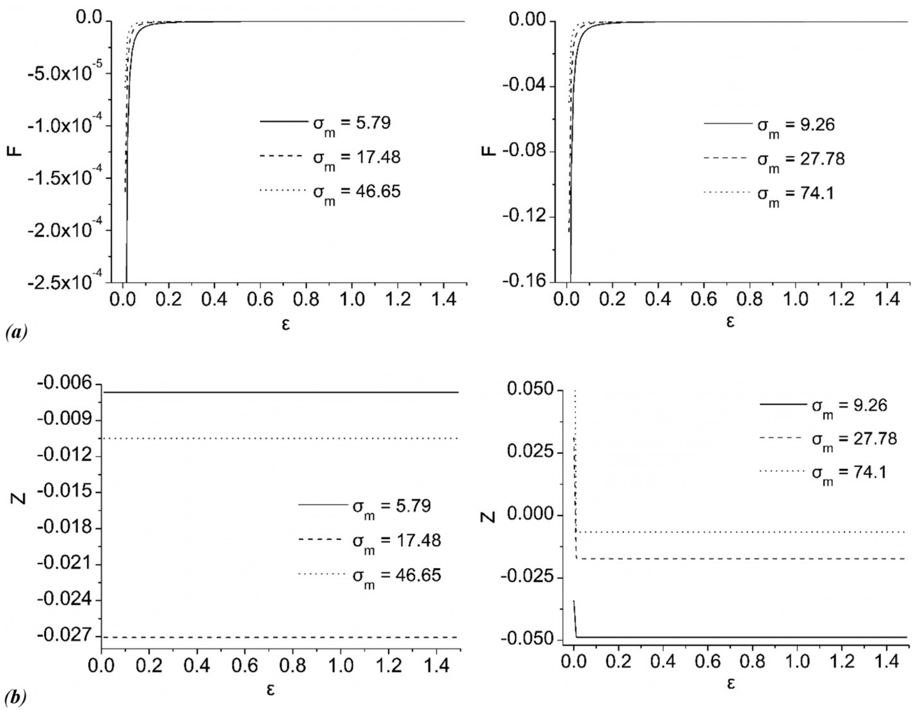

FIG. 6. Plot of (a) $F(\varepsilon)$ and (b) $Z(\varepsilon)$ for ferrofluid (left) and blood (right) using different values of the applied magnetic field (MHD micropolar Couette and Poiseuille flows with magnetic particles).

effectively retard fluid's velocity can also impede particles' microrotation. On the other hand, someone can argue that these wall forces can have an effect only on translational motion and the particles near the wall are able to freely spin. In this case, it is expected that the suspended particles will rotate at the same value with the walls' vorticity. The boundary condition of $\boldsymbol{\omega} = \boldsymbol{\Omega}$ at the walls is well-known among the researchers and it is widely used.[62,63]

It is clear that the two boundary conditions $\boldsymbol{\omega} = 0$ and $\boldsymbol{\omega} = \boldsymbol{\Omega}$ at the walls are completely opposite. For this reason, a linear relation is used, i.e., $\boldsymbol{\omega} = \delta\boldsymbol{\Omega}$, which constitutes a more general boundary condition and combines both $\boldsymbol{\omega} = 0$ and $\boldsymbol{\omega} = \boldsymbol{\Omega}$ through different values of the parameter δ. This parameter is a measure of the interactions between the particles and the wall which influences the rotation of the first. It varies as $0 \le \delta \le 1$; when $\delta = 0$, $\boldsymbol{\omega} = 0$ at the walls is implied, while the case of $\delta = 1$ implies that $\boldsymbol{\omega} = \boldsymbol{\Omega}$. In order to completely specify the effect of δ on micropolar flows, microrotation–vorticity difference $\omega_z - \Omega_z$ for different δ values will be calculated for the cases of the simple micropolar Couette and Poiseuille blood flows.

Table IV presents the results of microrotation–vorticity difference $\omega_z - \Omega_z$ for the simple micropolar Couette and Poiseuille blood flows for different values of δ. It is clear that as δ increases, the microrotation–vorticity difference decreases throughout the channel. When $\delta = 1$, then $\omega_z - \Omega_z = 0$, which is consistent with the profile of the corresponding Newtonian flow. These results reveal that as δ increases and tends to 1, $\omega_z \to \Omega_z$ which proves that the microrotation wall coefficient δ is an independent mechanism that affects $\omega_z - \Omega_z$ not only on the channel's walls, but throughout the whole channel's height leading to the simplification of the micropolar fluid equations down to the Navier–Stokes equations.

IV. SUMMARY AND CONCLUSIONS

In this study, the mechanics of micropolar MHD fluids with magnetic particles has been examined by reviewing the cases of vorticity $\boldsymbol{\Omega}$ and microrotation $\boldsymbol{\omega}$ (angular velocity of suspended particles) being equal. In general, when the difference $\boldsymbol{\omega} - \boldsymbol{\Omega}$ tends to zero (even when both $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ take non-zero values), the micropolar fluid equations are simplified to the classical Newtonian ones. In this manner, two classical examples of micropolar MHD fluids with magnetic particles were employed for the study, namely, blood and ferrofluid. These fluids were used in 2D Couette and Poiseuille flows. The effect of all dimensionless parameters that are relevant with the mathematical model was examined in terms of differentiating $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$. These dimensionless parameters were specifically calculated using the physical properties of the ferrofluid and blood in order to completely specify the cases that these fluids can be modeled as micropolar or Newtonian. The main findings of this study can be summarized as follows:

- Higher values of the micropolar effect parameter ε lead $\boldsymbol{\omega} = \boldsymbol{\Omega}$ for both fluids. This phenomenon is associated with higher values of the rotational viscosity μ_r (for definite values of dynamical viscosity μ and spin viscosity γ, which were specifically calculated for the ferrofluid and blood).
- Higher μ_r values are related to smaller values of spin relaxation time τ_s. This a contradiction to the common belief that higher values of μ_r (and in turn, higher values of ε) lead to higher micropolarity. Smaller values of τ_s imply that the equalization of microrotation and vorticity happens very quickly and any differences between the latter are not noticeable. Then, the micropolar flow equations are reduced to the classical model of the Navier–Stokes equations.

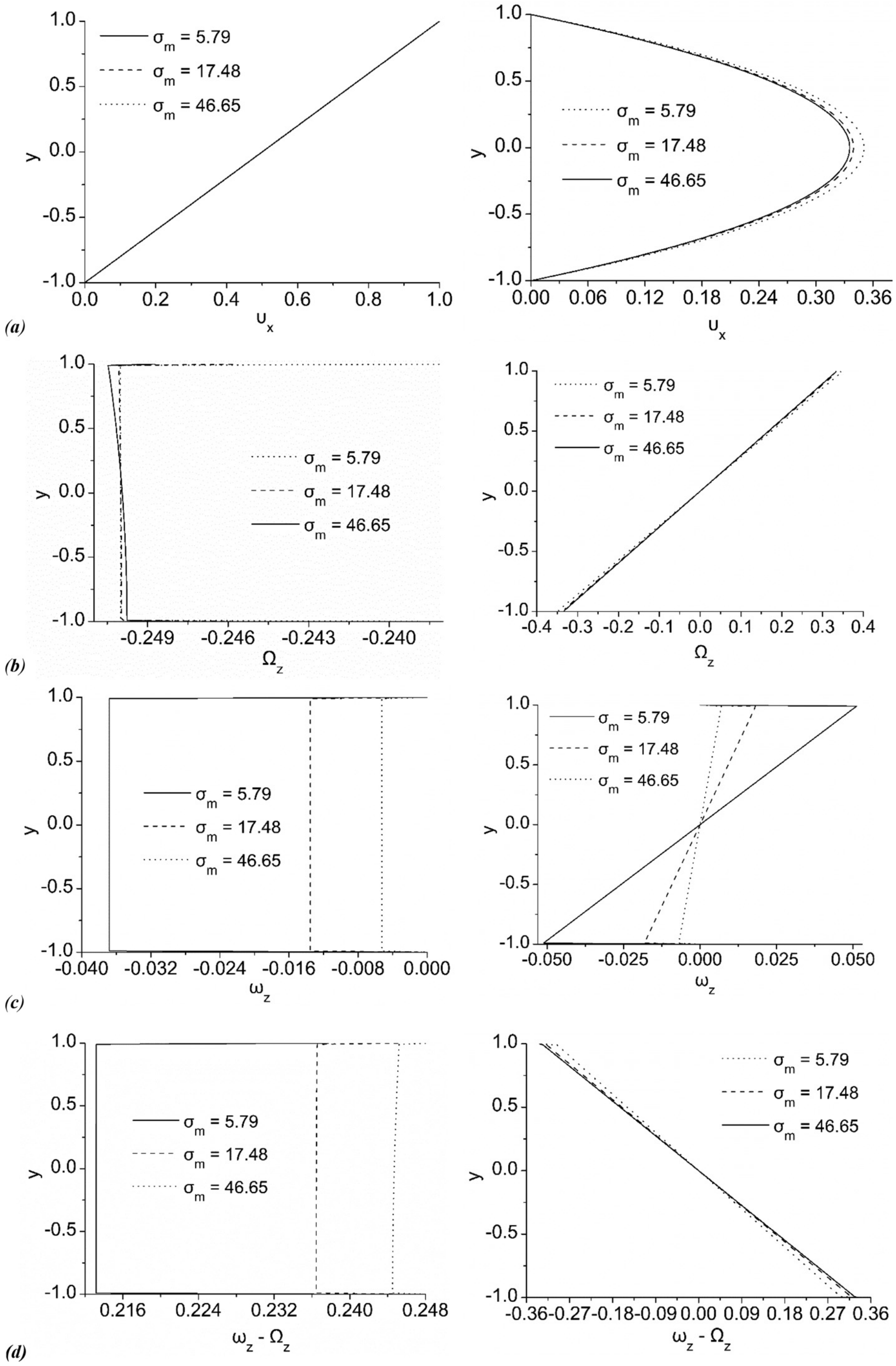

FIG. 7. Plots of (a) ferrofluid's velocity υ_x, (b) Vorticity Ω_z, (c) Microrotation ω_z and (d) Microrotation–Vorticity difference $\omega_z - \Omega_z$ using different values of the applied magnetic field.

- Higher values of the size effect parameter λ (which is proportional to the channel's height and inversely proportional to the spin viscosity γ) minimize $\boldsymbol{\omega} - \boldsymbol{\Omega}$. Higher λ values are associated with smaller γ values, which lead to smaller microrotation diffusion and allow the couple stress tensor to be ignored, minimizing $\boldsymbol{\omega} - \boldsymbol{\Omega}$.
- Channel's height is also important for the microrotation–vorticity difference. It was shown that blood's λ is small, especially when the channel's height takes the same value with the diameter of human arterioles, a situation that does not allow blood to be modeled as a Newtonian fluid. It is evident that λ should be calculated for all micropolar fluids before their simplification to Newtonian.

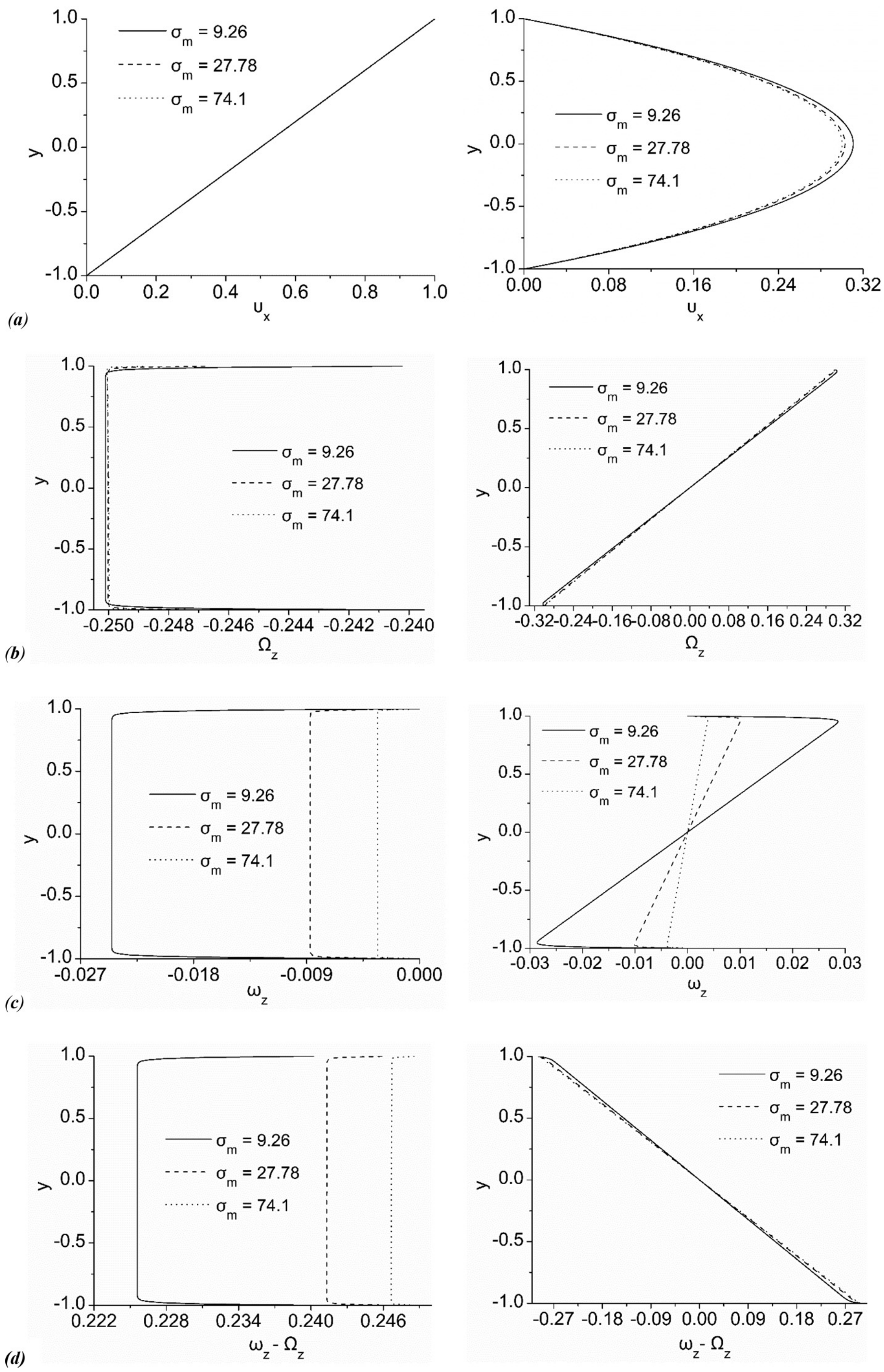

FIG. 8. Plots of (a) blood's velocity υ_x, (b) Vorticity Ω_z, (c) Microrotation ω_z, and (d) Microrotation–Vorticity difference $\omega_z - \Omega_z$ using different values of the applied magnetic field.

- The magnetic field does not seem to have any effect on $\boldsymbol{\omega} - \boldsymbol{\Omega}$ of micropolar fluids with small electrical conductivity (such as blood) when the magnetization of the particles is not considered. The micropolarity of these fluids stays the same as when no magnetic field is applied. The effect of the Lorentz force on the velocity of the micropolar fluid is very small to alternate vorticity through velocity and in turn to alternate microrotation.
- The magnetic field seems to differentiate microrotation from vorticity for both blood and ferrofluid when micromagnetorotation is considered, leading to higher micropolarity. This is a

TABLE IV. Results for microrotation–vorticity difference $\omega_z - \Omega_z$ for simple micropolar Couette and Poiseuille blood flows using different values of δ.

	$\delta = 0$		$\delta = 0.25$		$\delta = 0.5$		$\delta = 1$	
y	*Couette*	*Poiseuille*	*Couette*	*Poiseuille*	*Couette*	*Poiseuille*	*Couette*	*Poiseuille*
-1	1.54×10^{-1}	3.03×10^{-1}	1.28×10^{-1}	2.52×10^{-1}	9.54×10^{-2}	1.89×10^{-1}	0	0
-0.8	1.57×10^{-3}	3.09×10^{-3}	1.31×10^{-3}	2.57×10^{-3}	9.74×10^{-4}	1.93×10^{-3}	0	0
-0.6	1.61×10^{-5}	3.16×10^{-5}	1.33×10^{-5}	2.63×10^{-5}	9.94×10^{-6}	1.97×10^{-5}	0	0
-0.4	1.64×10^{-6}	3.23×10^{-6}	1.36×10^{-6}	2.68×10^{-6}	9.91×10^{-7}	2.01×10^{-6}	0	0
-0.2	1.68×10^{-7}	3.29×10^{-7}	1.39×10^{-7}	2.74×10^{-7}	9.85×10^{-8}	2.05×10^{-7}	0	0
0	1.42×10^{-7}	0	1.05×10^{-7}	0	9.73×10^{-8}	0	0	0
0.2	1.68×10^{-7}	-3.29×10^{-7}	1.39×10^{-7}	-2.74×10^{-7}	9.85×10^{-8}	-2.05×10^{-7}	0	0
0.4	1.64×10^{-6}	-3.23×10^{-6}	1.36×10^{-6}	-2.68×10^{-6}	9.91×10^{-7}	-2.01×10^{-6}	0	0
0.6	1.61×10^{-5}	-3.16×10^{-5}	1.33×10^{-5}	-2.63×10^{-5}	9.94×10^{-6}	-1.97×10^{-5}	0	0
0.8	1.57×10^{-3}	-3.09×10^{-3}	1.31×10^{-3}	-2.57×10^{-3}	9.74×10^{-4}	-1.93×10^{-3}	0	0
1	1.54×10^{-1}	-3.03×10^{-1}	1.28×10^{-1}	-2.52×10^{-1}	9.54×10^{-2}	-1.89×10^{-1}	0	0

standard result for ferrofluids, but it is not generally accepted for blood. The results of this paper prove that an externally applied magnetic field on a blood flow affects erythrocytes' microrotation, which in turn affects blood's velocity through vorticity, a phenomenon which is mentioned in many experimental studies and it cannot be explained only by the effect of the Lorentz force.

- Finally, the effect of microrotation boundary conditions $\boldsymbol{\omega} - \boldsymbol{\Omega}$ is discussed. The microrotation boundary conditions are associated with the microrotation wall parameter δ, which is a measure of the interactions between the particles and the wall. When $\delta = 0$, then $\boldsymbol{\omega} = 0$ at the walls, whereas in the case of $\delta = 1$, then $\boldsymbol{\omega} = \boldsymbol{\Omega}$ at the walls. It has been shown that for the simple micropolar Poiseuille and Couette blood flows, as $\delta \rightarrow 1$, microrotation tends to be equal to vorticity everywhere in the channel. For $\delta = 1$, $\boldsymbol{\omega} = \boldsymbol{\Omega}$ everywhere and microrotation is decoupled from velocity leading to the same flow solutions with the corresponding Newtonian ones.

In conclusion, an important issue of micropolar MHD fluids with magnetic particles (such as ferrofluids or blood) is the equalization of microrotation and fluid's vorticity. When the difference $\boldsymbol{\omega} - \boldsymbol{\Omega}$ tends to zero (even when both $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ take non-zero values), the micropolar fluid equations are simplified to the classical Navier–Stokes equations, a situation which brings all benefits associated with a simplified mathematical flow model (such as smaller computational cost and less computational time). It has been shown that higher values of the rotational viscosity (which are associated with smaller values of the spin relaxation time), higher values of channel's height and higher values of the microrotation wall parameter tend to equalize microrotation and vorticity. On the other hand, the spin viscosity and the micromagnetorotation (magnetic torque) are mechanisms that differentiate microrotation and vorticity. Lorentz force does not seem to have any noticeable effect of the microrotation–vorticity difference. Furthermore, numerical and experimental studies (such as studies for more complex blood and ferrofluid flow configurations) should be executed to fully identify the effect of these mechanisms on microrotation–vorticity difference.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Kyriaki-Evangelia Aslani: Conceptualization (lead); Formal analysis (lead); Investigation (lead); Methodology (lead); Validation (lead); Visualization (lead); Writing – original draft (lead); Writing – review & editing (equal). **Efstratios Tzirtzilakis:** Data curation (equal); Formal analysis (equal); Investigation (equal); Supervision (equal); Writing – review & editing (equal). **Ioannis E. Sarris:** Formal analysis (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX: VARIABLES APPEARING IN THE ANALYTICAL SOLUTIONS OF VELOCITY AND MICROROTATION

All constants used in the solution of linear velocity υ_z [Eq. (83)] and microrotation ω_y [Eq. (84)] for the micropolar Couette and Poiseuille flow with magnetic particles are as follows:

$$\xi_1 = 4\varepsilon\lambda^2(1 + \sigma_m) + \frac{Ha^2 - 4\varepsilon^2\lambda^2}{1 + \varepsilon}, \tag{A1}$$

$$\xi_2 = \frac{4\varepsilon\lambda^2(1 + \sigma_m)Ha^2}{1 + \varepsilon}, \tag{A2}$$

$$\xi_3 = \frac{2\varepsilon\lambda^2(1 + \sigma_m)}{1 + \varepsilon}, \tag{A3}$$

$$K = \frac{Ha^2 - 4\varepsilon^2\lambda^2}{2\varepsilon^2\lambda^2(1 + \sigma_m)}, \tag{A4}$$

$$\Lambda = \frac{1 + \varepsilon}{2\varepsilon^2\lambda^2(1 + \sigma_m)}, \tag{A5}$$

$$A = \frac{\sqrt{\xi_1 + \sqrt{{\xi_1}^2 - 4\xi_2}}}{\sqrt{2}}, \tag{A6}$$

$$B = \frac{\sqrt{\xi_1 - \sqrt{{\xi_1}^2 - 4\xi_2}}}{\sqrt{2}}, \tag{A7}$$

$$M = -1 + e^{2B}, \tag{A8}$$

$$N = -1 + e^{2A}, \tag{A9}$$

$$\Gamma = \frac{e^{-A+B}}{(-1 + e^{4B})\xi_2}, \tag{A10}$$

$$E = (e^{2A} - e^{2B})\xi_2, \tag{A11}$$

$$Z = \left(-1 + e^{2(A+B)}\right)\xi_2, \tag{A12}$$

$$H = e^{A} M \xi_3, \tag{A13}$$

$$C_1 = \frac{Ae^{A+B}\Gamma(-K + A^2\Lambda)MN\xi_3}{-Be^{A}\Gamma(E - Z)(-K + B^2\Lambda)M + Ae^{B}KN - A^3e^{B}\Lambda N}, \tag{A14}$$

$$C_2 = \frac{Ae^{A+B}\Gamma(-K + A^2\Lambda)MN\xi_3}{-Be^{A}\Gamma(E - Z)(-K + B^2\Lambda)M + Ae^{B}KN - A^3e^{B}\Lambda N}, \tag{A15}$$

$$C_3 = -\frac{Be^{2A}\Gamma(-K + B^2\Lambda)M^2\xi_3}{-Be^{A}\Gamma(E - Z)(-K + B^2\Lambda)M + Ae^{B}KN - A^3e^{B}\Lambda N}, \tag{A16}$$

$$C_4 = -\frac{Be^{2A}\Gamma(-K + B^2\Lambda)M^2\xi_3}{-Be^{A}\Gamma(E - Z)(-K + B^2\Lambda)M + Ae^{B}KN - A^3e^{B}\Lambda N}, \tag{A17}$$

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